Algebraic number theory with GP

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Documentation

- inside gp, `??refcard-nf`: list of functions with a short description.
- inside gp, `??`: Section 3.13: introduction and detailed descriptions of the functions.
- inside gp, `?10`: list of functions.
- inside gp, `?functionname`: short description of the function.
- inside gp, `??functionname`: long description of the function.

To record the commands we will type during the tutorial:

```
? \l TAN.log
```

Irreducibility

In GP, we describe a number field K as

$$K = \mathbb{Q}[x]/f(x)$$

where $f \in \mathbb{Z}[x]$ is a monic irreducible polynomial.

```
? f = x^4 - 2*x^3 + x^2 - 5;
? polisirreducible(f)
%2 = 1
```

GP knows cyclotomic polynomials:

```
? g = polcyclo(30)
%3 = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1
```

Algebraic numbers

Element of the quotient field $K = \mathbb{Q}[x]/f(x) = \mathbb{Q}(\alpha)$ where $f(\alpha) = 0$ are created using $Mod()$

```
? alpha = Mod(x,f);
? alpha^5
%4 = Mod(3*x^3-2*x^2+5*x+10, x^4-2*x^3+x^2-5)
? minpoly(alpha^5)
%5 = x^4-52*x^3+551*x^2-1250*x-3125
```

Interpretation: $\alpha^5 = 3\alpha^3 - 2\alpha^2 + 5\alpha + 10$.

```
? lift(alpha^15)
%6 = -1
```

We used `lift` (lift to $\mathbb{Q}[x]$) to make the output more readable.

polredbest

Sometimes we can find a simpler defining polynomial for the same number field by using `polredbest`:

```
? {h = x^5 + 7*x^4 + 22550*x^3 - 281686*x^2
  - 85911*x + 3821551};
? polredbest(h)
%8 = x^5 - x^3 - 2*x^2 + 1
```

Interpretation: $\mathbb{Q}[x]/h(x) \cong \mathbb{Q}[x]/(x^5 - x^3 - 2x^2 + 1)$.

nfinit

Most operations on number fields requires the structure of the ring of algebraic integers $\mathbb{Z}_K$, which is performed by the initialization function `nfinit`.

```
? K = nfinit(f);
```

`K` contains the data for the algebraic integers in the number field $K = \mathbb{Q}[x]/f(x)$.

```
? K.pol
%10 = x^4 - 2*x^3 + x^2 - 5
? K.sign
%11 = [2, 1]
```

K has signature $(2, 1)$: it has two real embeddings and one pair of conjugate complex embeddings.

Available information

```
? K.disc
%12 = -1975
? K.zk
%13 = [1,1/2*x^2-1/2*x-1/2,x,1/2*x^3-1/2*x^2-1/2*x]
? w = K.zk[2];
```

K has discriminant -1975, and its ring of integers is

$$\mathbb{Z}_K = \mathbb{Z}+\mathbb{Z}\frac{\alpha^2-\alpha-1}{2}+\mathbb{Z}\alpha+\mathbb{Z}\frac{\alpha^3-\alpha^2-\alpha}{2} = \mathbb{Z}+\mathbb{Z}w+\mathbb{Z}\alpha+\mathbb{Z}w\alpha.$$

Elements of a number field

We saw that we could represent elements of a number field as polynomials in α. We can also use linear combinations of the integral basis. We can switch between the two representations with `nfalgtobasis` and `nfbasistoalg`.

```
? nfalgtobasis(K,x^2)
%15 = [1, 2, 1, 0]~
```

Interpretation: $\alpha^2 = 1 \cdot 1 + 2 \cdot w + 1 \cdot \alpha + 0 \cdot w\alpha = 1 + 2w + \alpha$.

```
? nfbasistoalg(K,[1,1,1,1]~)
%16 = Mod(1/2*x^3 + 1/2, x^4 - 2*x^3 + x^2 - 5)
```

Interpretation: $1 + w + \alpha + w\alpha = \frac{\alpha^3+1}{2}$.

Elements of a number field: operations

We perform operations on elements with the functions `nfeltxxxx`, which accept both representations as input.

```
? nfeltmul(K,[1,-1,0,0]~,x^2)
%17 = [-1, 3, 1, -1]~
```

Interpretation: $(1 - w) \cdot \alpha^2 = -1 + 3w + \alpha - w\alpha$.

```
? nfeltnorm(K,x-2)
%18 = -1
? nfelttrace(K,[0,1,2,0]~)
%19 = 2
```

Interpretation: $N_{K/\mathbb{Q}}(\alpha - 2) = -1$, $\mathrm{Tr}_{K/\mathbb{Q}}(w + 2\alpha) = 2$.

Decomposition of primes

We can decompose primes with `idealprimedec`:

```
? dec = idealprimedec(K,5);
? #dec
%21 = 2
? [pr1,pr2] = dec;
```

Interpretation: $\mathbb{Z}_K$ has two prime ideals above 5, which we call $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

```
? pr1.f
%23 = 1
? pr1.e
%24 = 2
```

$\mathfrak{p}_1$ has residue degree 1 and ramification index 2.

Decomposition of primes

```
? pr1.gen
%25 = [5, [-1, 0, 1, 0]~]
```

$\mathfrak{p}_1$ is generated by 5 and $-1 + 0 \cdot w + \alpha + 0 \cdot w\alpha$, i.e. we have $\mathfrak{p}_1 = 5\mathbb{Z}_K + (\alpha - 1)\mathbb{Z}_K$.

```
? pr2.f
%26 = 1
? pr2.e
%27 = 2
```

$\mathfrak{p}_2$ also has residue degree 1 and ramification index 2.

Ideals

An arbitrary ideal is represented by its Hermite normal form (HNF) with respect to the integral basis. We can obtain this form with `idealhnf`.

```
? idealhnf(K,pr1)
%28 =
[5 3 4 3]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Interpretation: $\mathfrak{p}_1$ can be described as

$$\mathfrak{p}_1 = \mathbb{Z} \cdot 5 + \mathbb{Z} \cdot (w + 3) + \mathbb{Z} \cdot (\alpha + 4) + \mathbb{Z} \cdot (w\alpha + 3).$$

Ideals

```
? a = idealhnf(K,[23, 10, -5, 1]~)
%29 =
[260   0 228 123]
[  0 260 123 105]
[  0   0   1   0]
[  0   0   0   1]
```

We obtain the HNF of the ideal $\mathfrak{a} = (23 + 10w - 5\alpha + w\alpha)$.

```
? idealnorm(K,a)
%30 = 67600
```

We have $N(\mathfrak{a}) = 67600$.

Ideals: operations

We perform operations on ideals with the functions `idealxxxx`, which accept HNF forms, prime ideal structures (output of `idealprimedec`), and elements.

```
? idealpow(K,pr2,3)
%31 =
[25 15 21 7]
[ 0  5  2 4]
[ 0  0  1 0]
[ 0  0  0 1]
? idealnorm(K,idealadd(K,a,pr2))
%32 = 1
```

We have $\mathfrak{a} + \mathfrak{p}_2 = \mathbb{Z}_K$: the ideals $\mathfrak{a}$ and $\mathfrak{p}_2$ are coprime.

Ideals: factorisation

We factor an ideal into a product of prime ideals with `idealfactor`. The result is a two-column matrix: the first column contains the prime ideals, and the second one contains the exponents.

```
? fa = idealfactor(K,a);
? #fa[,1]
%34 = 3
```

The ideal $\mathfrak{a}$ is divisible by three prime ideals.

```
? [fa[1,1].p, fa[1,1].f, fa[1,1].e, fa[1,2]]
%35 = [2, 2, 1, 2]
```

The first one is a prime ideal above 2, is unramified with residue degree 2, and appears with exponent 2.

Ideals: factorisation

```
? [fa[2,1].p, fa[2,1].f, fa[2,1].e, fa[2,2]]
%36 = [5, 1, 2, 2]
? fa[2,1]==pr1
%37 = 1
```

The second one is $\mathfrak{p}_1$, and it appears with exponent 2.

```
? [fa[3,1].p, fa[3,1].f, fa[3,1].e, fa[3,2]]
%38 = [13, 2, 1, 1]
```

The third one is a prime ideal above 13, is unramified with residue degree 2, and appears with exponent 2.

Chinese remainders

We can use the Chinese remainder theorem with `idealchinese`:

```
? b = idealchinese(K,[pr1,2;pr2,1],[1,-1]);
```

We are looking for an element $b \in \mathbb{Z}_K$ such that $b = 1 \bmod \mathfrak{p}_1^2$ and $b = -1 \bmod \mathfrak{p}_2$.

```
? nfeltval(K,b-1,pr1)
%40 = 2
? nfeltval(K,b+1,pr2)
%41 = 1
```

We check the output by computing valuations: $v_{\mathfrak{p}_1}(b-1) = 2$ and $v_{\mathfrak{p}_2}(b+1) = 1$.

Chinese remainders with signs

We can compute the sign of real embeddings of b:

```
? nfeltsign(K,b)
%42 = [-1, 1]
```

We have $\sigma_1(b) < 0$ and $\sigma_2(b) > 0$, where σ_1, σ_2 are the two real embeddings of K.
We can ask `idealchinese` to compute an element that, in addition to the congruences, is totally positive:

```
? c = idealchinese(K,[[pr1,2;pr2,1],[1,1]],[1,-1]);
? nfeltsign(K,c)
%44 = [1, 1]
```

Indeed we have $\sigma_1(c) > 0$ and $\sigma_2(c) > 0$.

Dedekind zeta function

We can evaluated the Dedekind zeta function with `lfun`.

```
? L = nfinit(x^3-3*x-1);
? L.sign
%46 = [3, 0]
```

L is totally real.

```
? lfun(L,2)
%47 = 1.1722471496117109428809260096356285918
? q = bestappr(lfun(L,2)/Pi^6)
%48 = 8/6561
? lfun(L,2)/(Pi^6*q)
%49 = 1.0000000000000000000000000000000000000
```

$\zeta_L(2)$ is a rational multiple of π^6 (Siegel's theorem).

bnfinit

To perform computations of class groups and unit groups in a number field, we need a more expensive precomputation than the one from `nfinit`. We can perform this extra precomputation with `bnfinit` (b = Buchmann).

```
? K2 = bnfinit(K);
? K2.nf == K
%51 = 1
? K2.no
%52 = 1
```

K has a trivial group (no = class number).

```
? K2.reg
%53 = 1.7763300299706546701307646106399605586
```

We obtain an approximation of the regulator of K.

bnfcertify

The output of `bnfisprincipal` is a priori only correct under GRH (Generalised Riemann Hypothesis). We can unconditionally certify it with `bnfcertify`.

```
? bnfcertify(K2)
%54 = 1
```

The computation is now certified! If `bnfcertify` outputs `0`, it means we have found a counter-example to GRH (or more likely a bug in PARI/GP)!

bnfinit: units

```
? lift(K2.tu)
%56 = [2, -1]
? K2.tu[1]==nfrootsof1(K)[1]
%57 = 1
```

K has two roots of unity (tu = torsion units), ±1. We can also compute them with `nfrootsof1`.

```
? lift(K2.fu)
%58 = [1/2*x^2-1/2*x-1/2, 1/2*x^3-3/2*x^2+3/2*x-1]
```

The free part of $\mathbb{Z}_K^\times$ is generated by $\frac{\alpha^2-\alpha-1}{2}$ and $\frac{\alpha^3-3\alpha^2+3\alpha-2}{2}$ (fu = fundamental units).

bnfinit: analytic class number formula

```
? lfun(K,1+x+O(x^2))
%59 = 0.502284726052801113866176365679645651*x^-1
  + O(x^0)
? res = polcoeff(lfun(K,1+x+O(x^2)),-1)
%60 = 0.50228472605280111386617636567964565169
```

We compute an approximation of the residue of $\zeta_K(s)$ at $s = 1$.

```
? {2^K2.r1*(2*Pi)^K2.r2*K2.no*K2.reg/
    (K2.tu[1]*sqrt(abs(K2.disc))*res)}
%61 = 0.99999999999999999999999999999999999999
```

We numerically check the analytic class number formula.

Class group

```
? L = bnfinit(x^3 - x^2 - 54*x + 169);
? L.cyc
%63 = [2, 2]
```

$\mathcal{C}\ell(L) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

```
? L.gen
%64 = [[5,3,2;0,1,0;0,0,1], [5,4,3;0,1,0;0,0,1]]
```

Generators of the class group, given as ideals in HNF form.

Testing whether an ideal is principal

We can test whether an ideal is principal with `bnfisprincipal`:

```
? pr = idealprimedec(L,13)[1]
? [dl,g] = bnfisprincipal(L,pr);
? dl
%67 = [1, 0]~
```

`bnfisprincipal` expresses the class of the ideal in terms of the generators of the class group (discrete logarithm). Here, the ideal `pr` is in the same class as the first generator. In particular, the ideal is not principal, but its square is.

Testing whether an ideal is principal

```
? g
%68 = [0, 1/5, 2/5]~
? {idealhnf(L,pr) == idealmul(L,g,
    idealfactorback(L,L.gen,dl))}
%69 = 1
```

The second component of the output of `bnfisprincipal` is an element $g \in L$ that generates the remaining principal ideal. (idealfactorback = inverse of idealfactor = $\prod_i \texttt{L.gen[i]}^{\texttt{dl[i]}}$)

Computing a generator of principal ideal

We know that `pr` is a 2-torsion element; let's compute a generator of its square:

```
? [dl2,g2] = bnfisprincipal(L,idealpow(L,pr,2));
? dl2
%71 = [0, 0]~
```

The ideal is indeed principal (trivial in the class group).

```
? g2
%72 = [1, -1, -1]~
? idealhnf(L,g2) == idealpow(L,pr,2)
%73 = 1
```

`g2` is a generator of `pr`^{2}.

Application: bnfisintnorm

We can use these functionalities to find solutions in $\mathbb{Z}_K$ of norm equations with `bnfisintnorm`:

```
? bnfisintnorm(L,5)
%74 = []
```

There is no element of norm 5 in $\mathbb{Z}_L$.

```
? bnfisintnorm(L,65)
%75 = [x^2 + 4*x - 36, -x^2 - 3*x + 39, -x + 2]
```

There are three elements of $\mathbb{Z}_L$ of norm 65, up to multiplication by elements of $\mathbb{Z}_L^\times$ with positive norm.

Expressing a unit in terms of the generators

```
? u = [0,2,1]~;
? nfeltnorm(L,u)
%77 = 1
```

We have found a unit $u \in \mathbb{Z}_L^\times$.

```
? bnfisunit(L,u)
%78 = [1, 2, Mod(0, 2)]~
? lift(L.fu)
%79 = [x^2 + 4*x - 34, x - 4]
? lift(L.tu)
%80 = [2, -1]
```

We express it in terms of the generators with `bnfisunit`:
$u = (\alpha^2 + 4\alpha - 34) \cdot (\alpha - 4)^2 \cdot (-1)^0$.

Large fundamental units

By default, `bnfinit` only computes fundamental units if they are not too large.

```
? M = bnfinit(x^2-3019);
? M.fu
  ***   at top-level: M.fu
  ***                   ^--
  *** _.fu: missing units in bnf.
```

We can force the computation of fundamental units with `bnfinit(,1)`.

```
? M = bnfinit(x^2-3019,1);
? lift(M.fu)
%83 = [2138951880537520985460710555927255657066690
  871236169789*x - 1175256254165994101844252641552
  37539460392094825860314330]
```

Questions ?

Have fun with GP!