Polylogarithms, multiple zeta values and K-theory using PARI/GP

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Atelier Grenoble 2016, 15.1.16

Motivation: Dirichlet's Class Number Formula

Dirichlet's CNF connects classical arithmetic data of F ($\#$ field):

1) units 2) class group 3) Dedekind zeta value

"visible" in the residue at the Dedekind zeta function at $s = 1$ or, via functional eqn, in Taylor expansion around $s = 0$

$$\zeta_F^*(0) = -\frac{h_F\, R_F}{w_F} = -\frac{|\mathcal{C}\ell(\mathcal{O}_F)|\ \ covol(\text{``}\log|\mathcal{O}_F^\times|\text{''})}{|\text{roots of unity in } \mathcal{O}_F|}.$$

Any *higher analogue* of Dirichlet's CNF should connect "higher arithmetic" data:

1) higher units 2) higher class group 3) Dedekind zeta value at integers $m > 1$ or rather at $1 - m$. Could ask for a formula of type

$$\zeta_F^*(1-m) \stackrel{?}{=} \lambda\, \frac{|\mathcal{C}\ell_m(\mathcal{O}_F)|\ covol\big(\text{``}\log_m(m\text{-th higher units})\text{''}\big)}{|m\text{-th higher roots of unity for } \mathcal{O}_F|},$$

with λ "simple" and understood (explicit power of π times a rational number).

Generalization: K-groups

Right setting for generalization: "algebraic K-groups" of $\mathcal{O}_F$, denoted $K_n(\mathcal{O}_F)$ (abelian groups of finite rank).
One has 2) $K_0(\mathcal{O}_F) = \mathbf{Z} \oplus \mathcal{C}\ell(\mathcal{O}_F)$ and 1) $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$.

and Lichtenbaum's Conjecture asks for a formula

$$\zeta_F^*(1-m) \stackrel{?}{=} \lambda \frac{|K_{2m-2}(\mathcal{O}_F)_{\mathrm{tors}}|}{|K_{2m-1}(\mathcal{O}_F)_{\mathrm{tors}}|} \times covol\big(\log_m K_{2m-1}(\mathcal{O}_F)\big) .$$

Problem: $K_{2m-1}(\mathcal{O}_F)$ highly abstract.

Remedy: concrete candidates by Bloch ($m=2$), Zagier ($m>2$).
Experimentally accessible!

E.g. higher unit group for $m=2$ is given as a subquotient of $\mathbf{Z}[F]$; crucial map:

$$\begin{aligned} \partial_2^F : \mathbf{Z}[F] &\longrightarrow F^\times \otimes F^\times \\ [x] &\mapsto x \otimes (1-x) \end{aligned}$$

Key object: $\ker \partial_2^F$.

Higher analogs for odd-indexed K-groups

Higher unit group for general m via $\ker \partial_m^F$ with crucial map

$$\begin{array}{rcll} \partial_m^F : \mathbf{Z}[F] & \longrightarrow & F^\times \otimes \cdots \otimes F^\times & \quad (m \text{ copies}) \\ [x] & \mapsto & x \otimes \cdots \otimes x \otimes (1-x) & \end{array}$$

(i.e. like for $m = 2$ but with tensor factor x repeated $m - 1$ times).

Good fct on $\ker \partial_m^F$: single-valued variant $\mathcal{L}_m(z)$ of m-logarithm

$$Li_m(z) = \sum_{k \geq 1} \frac{z^k}{k^m}, \qquad (|z| < 1)$$

(pari notation for $\mathcal{L}_m(z)$ is `polylog(m,z,3)`, say) $\rightsquigarrow$ our $\log_m$!
Evaluating $\mathcal{L}_m$ on $\ker \partial_m^F$ gives lattice! (ignore inductive condition)
with covolume related to $\zeta_F^*(1 - m)$.

Experimental results and conjectures

In many cases (degree + discriminant not too large) get non-zero covolume for small m.

Benefits: 1) can formulate combined Lichtenbaum-Zagier conjecture (again, small m), making “missing factor” λ precise; 2) glimpse into spectrum of orders of *even-indexed* K-groups.

Examples: 0. For most fields of small discriminant relative to degree get only boring factors.
1. For F of signature $[4, 1]$ and discriminant $-7^4 \cdot 43$ we predict $79 \mid K_6(\mathcal{O}_F)$.
2. For F of signature $[2, 2]$ and (modest) discriminant 229^2 predict two prime factors $131 \cdot 138191 \mid K_{10}(\mathcal{O}_F)$.

Problem: For $m > 4$ often non-maximal rank, hence covolume 0. Need to run procedure many times with different parameters. Bottleneck: `qflll` for 500×1500 matrices with L2-norm 10^{250}.

Proven results

(with **Karim**, around 2000): implemented "Tate's algorithm" in PARI to get *proven* results for K_2 (even its structure) in hundreds of cases. Thus corroborated all except a handful of cases of discriminant > -5000 for imaginary quadratic fields.

Problem: Large prime factors p dividing $|K_2\mathcal{O}_F|$ require understanding of class group of cyclotomic p-extension of F.

(with **Philippe Elbaz-Vincent, C. Soulé**, around 2002): implemented "Voronoi algorithm" originally in PARI; instrumental for proving the longstanding problems of determining $K_5(\mathbf{Z})$ and $K_6(\mathbf{Z})$.

Multiple polylogarithms and multiple zeta values

Generalise polylogs to many parameters ("multiple polylogs"):

$$Li_{m_1,m_2}(z_1, z_2) = \sum_{k_1 > k_2 > 0} \frac{z_1^{k_1}}{k_1^{m_1}} \frac{z_2^{k_2}}{k_2^{m_2}}$$

Re-specialise to $z_i = 1$, giving $\zeta(m_1, m_2)$, a "multiple zeta value". These are expected to be transcendental numbers; they constitute an important class of periods with lots of linear relations between them. Now also implemented as `zetamult` in PARI (**Henri**, with support from Karim)—beautiful playground for experiments!

Perhaps surprising: there is even a relationship with modular forms (for $\mathrm{SL}_2(\mathbf{Z})$), via period polynomials! E.g. cusp form Δ implies

$$28\zeta(9,3) + 150\zeta(7,5) + 168\zeta(5,7) = \frac{5197}{691}\zeta(12)\,.$$