Local behaviour of Galois representations

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Natural to ask, **when does $\rho_f|_{G_p}$ split?**

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f is non-CM $\overset{??}{\Longrightarrow} \rho_f|_{G_p}$ is non-split.

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For $p = 3$, every member of a non-CM family* $\mathcal{H}_f$ is non-split if

- condition $(C1)$
- condition $(C2)$

are satisfied.

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Remark

The spaces $\widetilde{\mathrm{Cl}}_H$, $\tilde{E}$, $\widetilde{U}_p$ and $\widetilde{U}_{p,0}$ are all $\mathbb{F}_p[G_{\mathbb{Q}}]$-modules.

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For a finite dimensional $\mathbb{F}_p[G_{\mathbb{Q}}]$-module V, let
$V^{\mathrm{Ad}} :=$ sum of all J-H factors isomorphic to W_0.

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When $N = 118$, hypothesis $(C2)$ fails.

We give an alternative argument to deal with such cases.

Conditions $(C1)$ and $(C2)$

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Note that W_0 is an irreducible $\mathrm{GL}_2(\mathbb{F}_p)$-module, while the conditions (C1) and (C2) are over $\mathrm{PGL}_2(\mathbb{F}_p)$. This is because scalars act trivially on W_0.

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This works well in computations as the order of $\mathrm{GL}_2(\mathbb{F}_p)$ is 48, where as $\mathrm{PGL}_2(\mathbb{F}_p)$ is 24!

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This uses PARI/GP!

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- $e \in 1 + \mathfrak{P}^n$, for $n \leqslant 2$

Elliptic curves satisfying conditions $(C1)$ and $(C2)$

A_f	Δ_{A_f}	(a,b) for A_f	$\lvert\mathrm{Cl}_H\rvert$	e	e lies in
89.a1	-89	$(-1323, 28134)$	2	$e_4^{-2}e_5^2e_6^{-2}e_7e_8^2e_9^{-2}$	$1+\mathfrak{P}_2$
155.a1	$-5^5\cdot 31$	$(12528, 443664)$	$2\cdot 3$	$e_2^4e_3^4e_4^4e_5^6e_7^{-4}e_8^4e_9^2$	$1+\mathfrak{P}_1$
155.b1	$-5^2\cdot 31$	$(-1323, -65178)$			
158.b1	$2^2\cdot 79$	$(-4563, 111726)$	$2\cdot 3$	$-e_1^2e_2e_3e_5e_6^{-1}$	$1+\mathfrak{P}_8^2$
158.c1	$2^{20}\cdot 79$	$(-544347, 153226998)$			
158.e2	$2\cdot 79^2$	$(-11691, 416934)$			
⋮	⋮			⋮	
994.b2	$2^2\cdot 7^{10}\cdot 71$	$(-1509219, -324105570)$	$2\cdot 3^4\cdot 13^3$	$-e_5e_6e_7^{-2}e_8^2e_9^{-2}e_{10}^{-2}$	$1+\mathfrak{P}_1$
994.e2	$2\cdot 7^2\cdot 71^2$	$(-27243, -711450)$			

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- Alternative condition $(C2')$: This involves showing that a particular totally ramified $\mathbb{Z}/3$-extension K_3 over $\mathbb{Q}_3$ is distinct from the cyclotomic $\mathbb{Z}/3$-extension over $\mathbb{Q}_3$.

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- Condition $(C2)$ fails!
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Checking the alternative condition includes explicitly computing the norm subgroup corresponding to $K_3/\mathbb{Q}_3$. This uses PARI-GP extensively.

Thank You.