Congruences between Modular forms of level 1

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Definition

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be an eigenform of level 1 and weight k. The $p-$adic slope of f is $\alpha(f) = v_p(a_p)$.

It's the valuation of the eigenvalue of the operator associated to p.

Coleman's theorem

Theorem (Coleman)

Let f an eigenform for $\mathrm{SL}_2(\mathbb{Z})$ of weight k and slope α. There exists $r \in \mathbb{N}$ such that: for every $k' \in \mathbb{N}$, $k' > \alpha + 1$ and for every $m \in \mathbb{N}$ if

$$k \equiv k' \mod p^{m+r-1}(p-1)$$

then there exist a modular form f' of level 1, weight k' and slope α such that

$$f \equiv f' \mod p^m.$$

$p-$adic modular forms, definition

Definition (Serre)

Let M_k be the space of classical modular forms of level 1 and weight k. An expansion

$$f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[[q]]$$

is a $p-$adic modular form if there exist a sequence of classical modular forms $f_i \in M_{k_i}$ such that $v_p(f - f_i) \to \infty$ when $i \to \infty$.

$p-$adic modular forms, weight

Theorem (Serre)

Let

$$X = \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{Z}_p^*) \cong \frac{\mathbb{Z}}{(p-1)\mathbb{Z}} \times \mathbb{Z}_p$$

If f is a $p-$adic modular form and if f_i is the sequence of classical forms of weight k_i converging to f, then there exist a unique $\kappa \in X$ such that the sequence k_i converges to κ. This is independent of the choice of the sequence f_i and we call it the weight of f.

$p-$adic analytic families

Let $Y = \{m\} \times B(c,\delta) \subset \frac{\mathbb{Z}}{(p-1)\mathbb{Z}} \times \mathbb{Z}_p$ be the ball of centre c such that $v_p(c-k) > \delta$.

Definition

A $p-$adic analytic family of modular forms is a formal power series $\sum_n F_n(\kappa)q^n$ where every $F_n : B(c,\delta) \to \mathbb{Q}_p$ is a $p-$adic analytic function, and such that for $k_0 \in \mathbb{Z} \cap Y$, the expression $\sum F_n(k_0)q^n$ is the expansion of a classical form of weight k_0 and level 1.

Congruences

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Theorem (D. - Kiming)

Let $p = 5$*. Let* $k_0 \in 4\mathbb{Z}$ *et* $v_5(k_0 - 8) \leq 1$*. There exists a* $p-$*adic faimily of modular forms* $f(k) = \sum_n a_n(k)q^n$ *such that* $f(k_0)$ *is a* $p-$*adic eigenform of level* 1*, weight* k_0 *and slope* $\alpha = 1 + v_5(k_0 - 8)$*.*

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Let $p = 5$. Let $k_0 \in 4\mathbb{Z}$ et $v_5(k_0 - 8) \leq 1$. There exists a $p-$adic faimily of modular forms $f(k) = \sum_n a_n(k)q^n$ such that $f(k_0)$ is a $p-$adic eigenform of level 1, weight k_0 and slope $\alpha = 1 + v_5(k_0 - 8)$. Moreover, if k_0 and k_1 have this property, we have that

$$v_5(a_n(k_0) - a_n(k_1)) \geq v_5(k_0 - k_1) + \inf\{1, 2 - v_5((k_0 - 8)(k_1 - 8))\}$$

Example

Let $k_0 = 12$. Then the only eigenform of weight 12 is Δ; the 5−adic slope of Δ is 1, as $1 + v_5(12 - 8) = 1$.

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Let $k_1 = 112$. Then there is only one eigenform f_1 of weight 112 and slope $1 + v_5(112 - 8) = 1$.
Since $112 \equiv 12 \mod 5^2$, it follows from the theorem that

$$\Delta \equiv f_1 \mod 5^3$$

.

Proof

The space of (overconvergent) $p-$adic modular forms of (tame) level 1 and weight k, denoted M_k has a structure of $p-$adic Banach Space.

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On M_k acts Atkin's U_p operator, that has a characteristic power series

$$P_k(T) = \sum C_n T^n$$

.

Coleman's trick

Crucial Idea: construct an operator U^* that interpolates the action of Atkin's U operator on the space of $p-$adic modular forms of varying weight κ.

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The characteristic series of the U^* operator is then a **two-variable** power series

$$P(\kappa, T) = \sum C_n(\kappa) T^n,$$

.

Newton Polygon

We study $P(\kappa, T)$ with **Newton polygons**. The aim is to prove that the first segment of the Newton polygon has length 1. This imply the uniqueness of the eigenform of lowest slope.

Koike's formula

To explicitely compute the values of $P(\kappa, T)$ we need Koike's formula: a $p-$adic version of the Selberg trace formula.

Theorem (Koike)

Let $\gamma(t)$ be the $p-$adic unit root of the equation $x^2 - tx + p^n$. Let

$$B(n, \kappa) = \sum_{\substack{0 \le t < 2\sqrt{p^n} \\ (t,p)1}} H(t^2 - 4p^n) \frac{\gamma(t)^\kappa}{\gamma(t)^2 - p^n}. \tag{1}$$

Then

$$P(\kappa, T) = \exp\left(\sum_n B(n, \kappa) \frac{T^n}{n}\right)$$

Hurwitz class number

Where $H(D)$ is the Hurwitz class number, that is, the number of equivalence classes of binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant D.

Koike

The advantage is that with careful computations, one can compute the coefficients with κ as a variable:

```
\\syntax: koike(p,accuracy,k,n)
? P=koike(5,1000,k,4);
? lift(Mod(P,5^4))
% 1 + (310 + 855*k + 400*k^2 + 2125*k^3 + 2500*k^5)*T +
(2500*k + 625*k^2 + 2500*k^3 + 625*k^4)*T^2 + O(T^4)
```

Thank you for your attention!