Using approximate functional equations to build L functions

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Example : elliptic curves

Consider an elliptic curve $E/\mathbb{Q}$ of conductor N and root number $\varepsilon = -1$. The associated modular form

$$f = \sum a_n q^n, q = e^{2i\pi z}. \tag{1}$$

satifies $Wf = -f$ and vanishes at $q = e^{-\frac{2\pi}{\sqrt{N}}}$,

$$f(q) = 0 = q + a_2 q^2 + a_3 q^3 + \dots. \tag{2}$$

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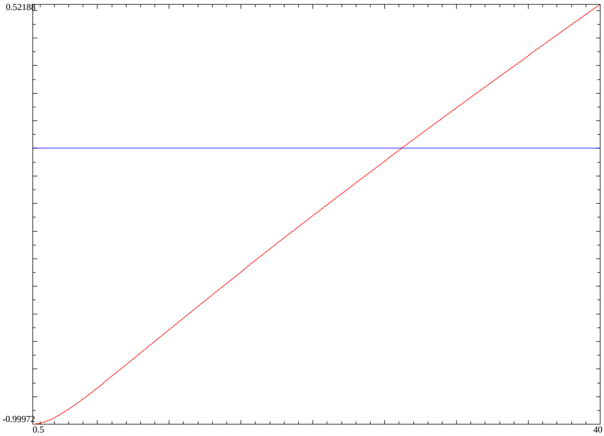
By Hasse, $|a_n| \leqslant \lfloor \sigma_0(n)\sqrt{n} \rfloor \leqslant n$, so the equality is possible only if

$$q \leqslant \sum_{k \geqslant 2} nq^n = \frac{q}{(1-q)^2} - q = \frac{q^2(2-q)}{(1-q)^2}. \qquad (3)$$

```
b(q)=q*(2-q)/(1-q)^2;
plot(N=.5,40,b(exp(-2*Pi/sqrt(N))))
```



```
b(q)=q*(2-q)/(1-q)^2;
plot(N=.5,40,b(exp(-2*Pi/sqrt(N))))
```

```
solve(N=.5,40,b(exp(-2*Pi/sqrt(N)))-1)
26.181852174699964975652391885916899331
```

Theorem

If an elliptic curve has rank $r \geqslant 1$, its conductor satisfies $N \geqslant 27$.

Least conductor : generalize

Degree 2 L-function $L(s)$, one gamma factor $\Gamma_{\mathbb{C}}(s)$, conductor N, weight k and sign ε. The (symmetrized) inverse Mellin transform

$$F(x) = e^{\frac{x}{2}} \sum a_n e^{-\frac{2\pi}{\sqrt{N}} e^x n} \tag{4}$$

satisfies

$$F(x) = \varepsilon \overline{F}(-x). \tag{5}$$

In particular for all $0 < y < \frac{\pi}{2}$, $F(iy) - \epsilon F(-iy) = 0$.
Let $t = e^{iy\frac{k}{2}}$, $q = e^{-\frac{2\pi}{\sqrt{N}} e^{iy}}$, one must have

$$\sum_{n \geqslant 1} a_n (tq^n - \epsilon \overline{tq^n}) = 0$$

Least conductor : generalize

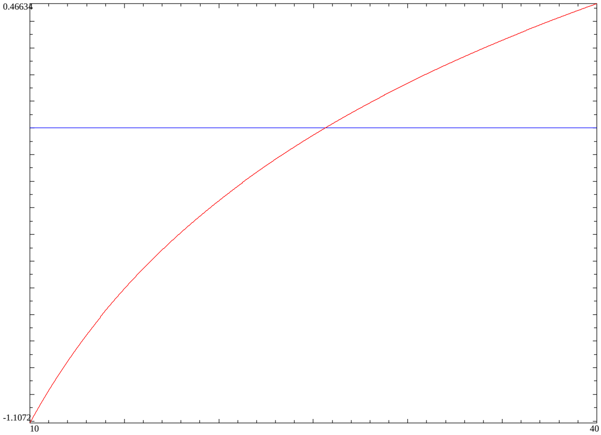
Using

- $|a_n| \leqslant \lfloor \sigma_0(n) n^{\frac{k-1}{2}} \rfloor \leqslant \sqrt{3} n^{\frac{k}{2}}$
- $\sum_{n>K} n^{\frac{k}{2}} |q|^n \leqslant \frac{\sqrt{K+1}^k}{1-\frac{k}{2\alpha_y (K+1)}} |q|^K = B_K(q)$ if $2\alpha_y(K+1) > k$

one must have ($t = e^{iy\frac{k}{2}}$, $q = e^{-\frac{2\pi}{\sqrt{N}} e^{iy}}$)

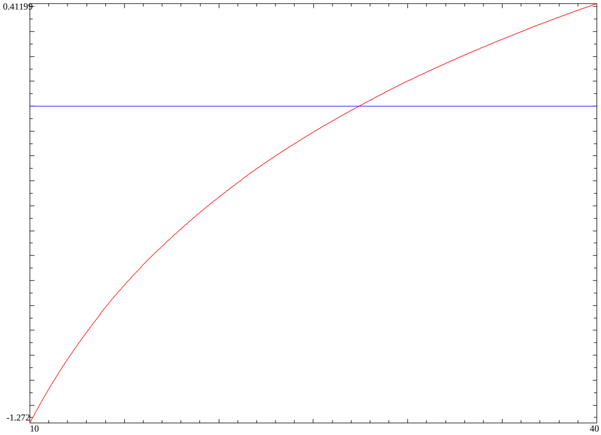
$$|tq - \epsilon \overline{tq}| \leqslant \sum_{n=2}^{K} \lfloor \sigma_0(n) n^{\frac{k-1}{2}} \rfloor |tq^n - \epsilon \overline{tq^n}| + 2\sqrt{3} B_K(q)$$

Odd elliptic curves



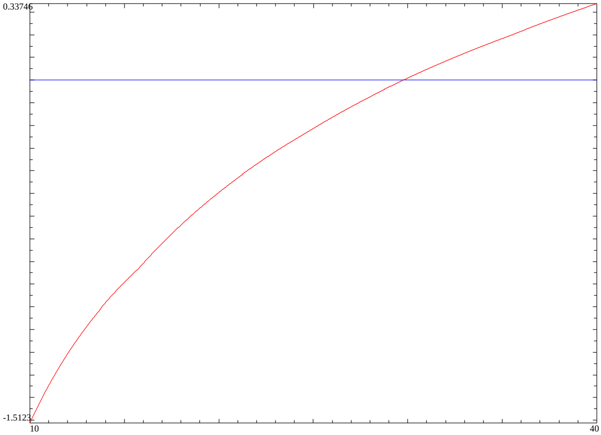
$y = 0.1 \Rightarrow N \geqslant 25.63$ better !

Odd elliptic curves



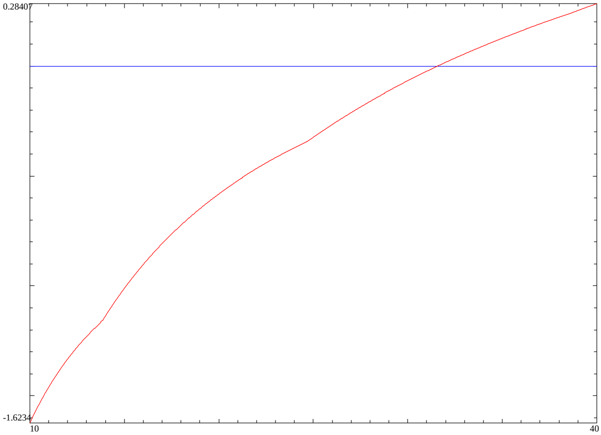
$y = 0.2 \Rightarrow N \geqslant 27.40$. better !

Odd elliptic curves



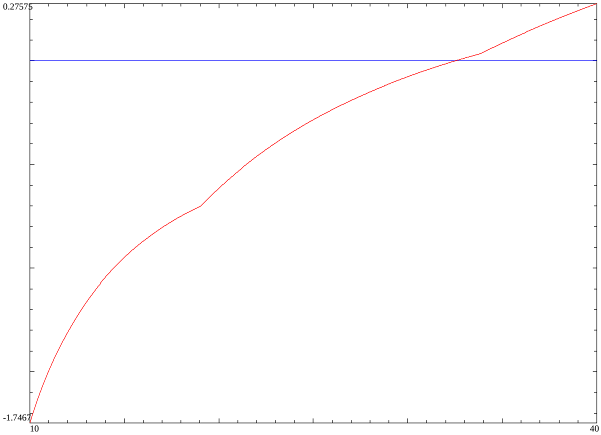
$y = 0.3 \Rightarrow N \geqslant 29.77$ .. better !

Odd elliptic curves



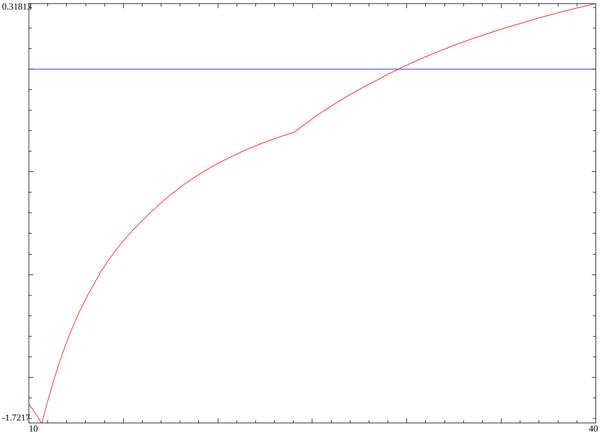
$y = 0.4 \Rightarrow N \geqslant 31.58$ ... better !

Odd elliptic curves



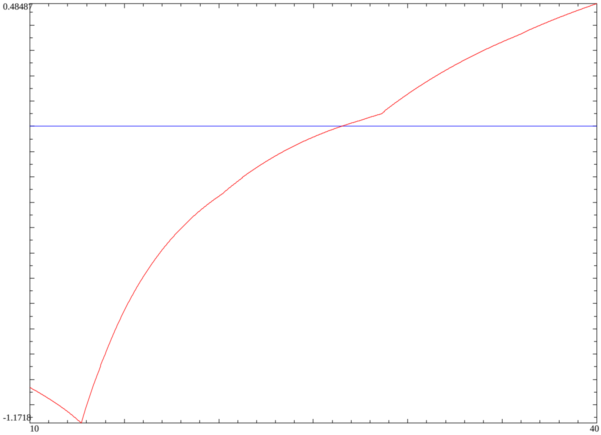
$y = 0.5 \Rightarrow N \geqslant 32.53$ .... better !

Odd elliptic curves



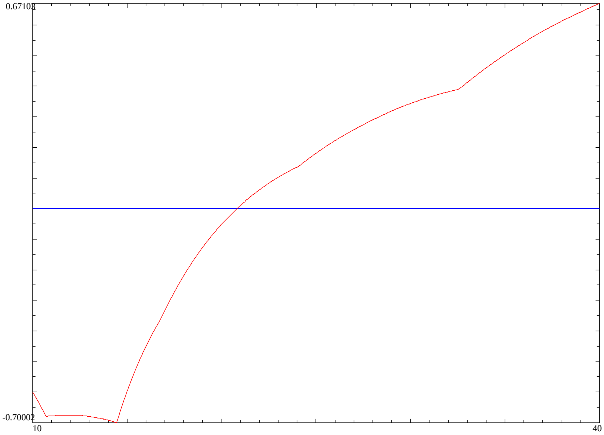
$y = 0.6 \Rightarrow N \geqslant 29.57$ ... too bad !

Odd elliptic curves



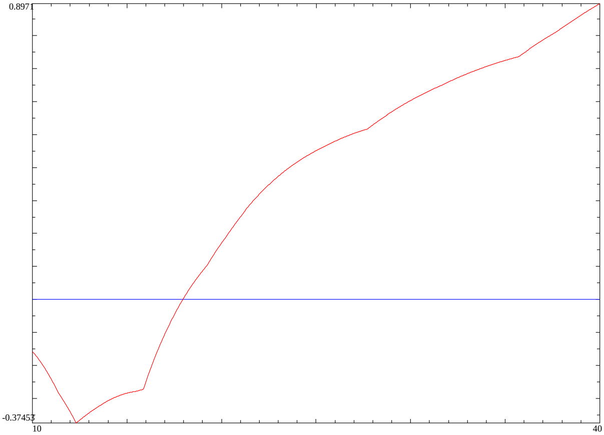
$y = 0.7 \Rightarrow N \geqslant 26.48$ .. and worse

Odd elliptic curves



$y = 0.8 \Rightarrow N \geqslant 20.81$. and worse

Odd elliptic curves



$y = 0.9 \Rightarrow N \geqslant 17.95$ and worse

- Equation $F(0) = 0 \rightsquigarrow N \geqslant 27$.
- Equation $F(iy) + F(-iy) = 0, y = .5 \rightsquigarrow N \geqslant 33$

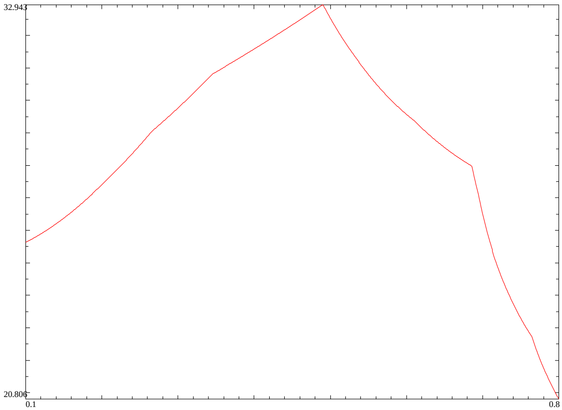
Theorem

If an elliptic curve has rank $r \geqslant 1$, its level satisfies $N \geqslant 33$.

(was $N \geqslant 27$ previously)

Automatic results

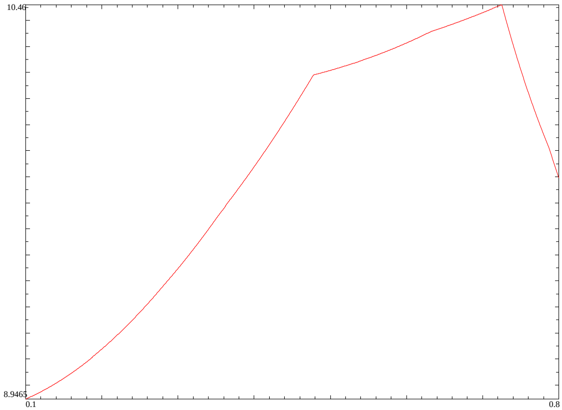
Plot on y for the least value of N satisfying inequality.



$$k = 2,\ \varepsilon = -1 \quad N \geqslant 33$$

Automatic results

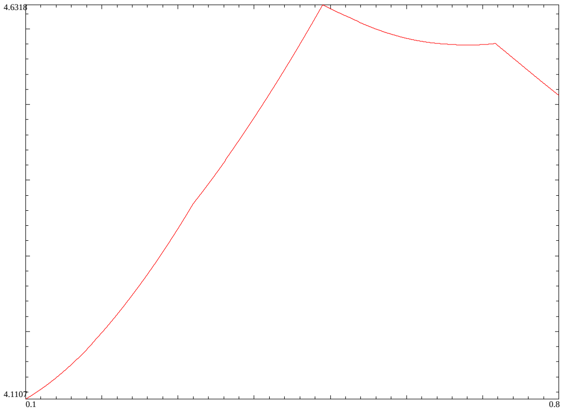
Plot on y for the least value of N satisfying inequality.



$$k = 2,\ \varepsilon = +1 \quad N \geqslant 11$$

Automatic results

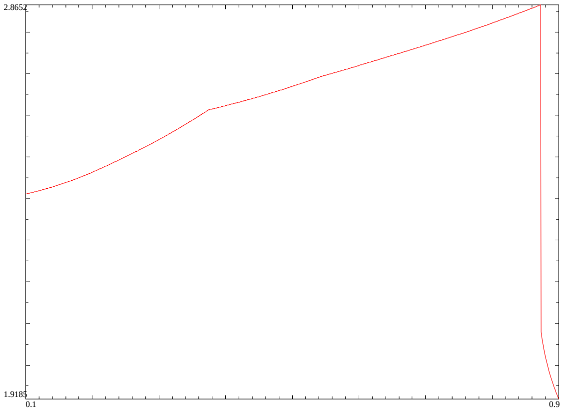
Plot on y for the least value of N satisfying inequality.



$$k = 4,\ \varepsilon = +1 \quad N \geqslant 5$$

Automatic results

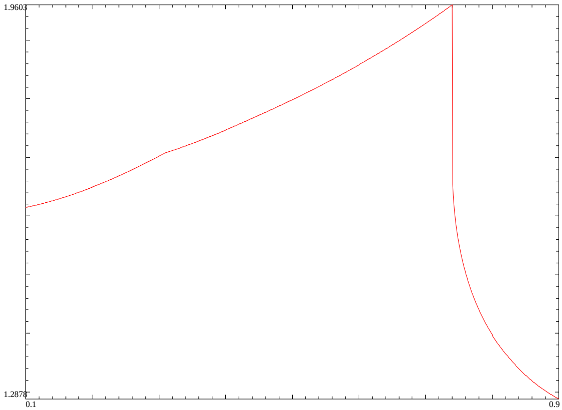
Plot on y for the least value of N satisfying inequality.



$$k = 6,\ \varepsilon = +1 \quad N \geqslant 3$$

Automatic results

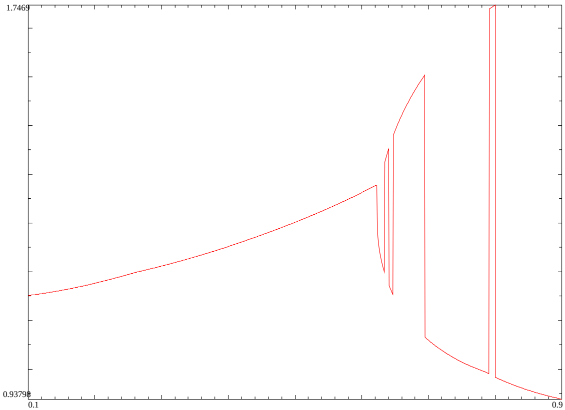
Plot on y for the least value of N satisfying inequality.



$$k = 8,\ \varepsilon = +1 \quad N \geqslant 2$$

Automatic results

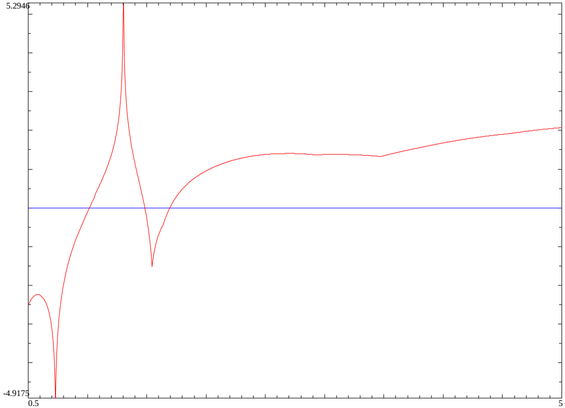
Plot on y for the least value of N satisfying inequality.



$k = 10$, $\varepsilon = +1$ ...hum

Bifurcation

the ratio is not monotonous and crosses the 1-axis several times.



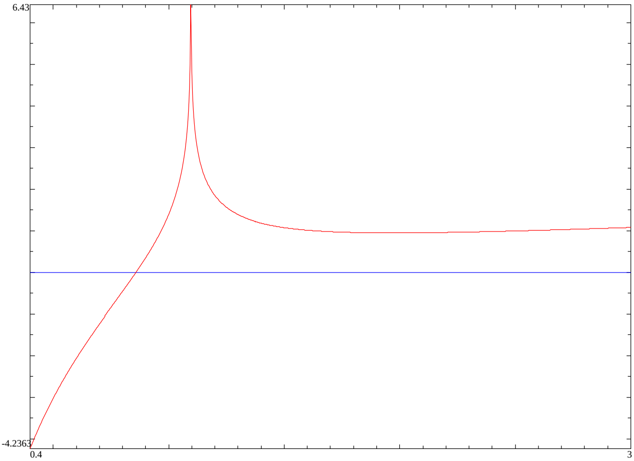
Still the result $N > 1.7$ is correct, combining several plots.

Results for degree 2

weight k and level N, root number ϵ.

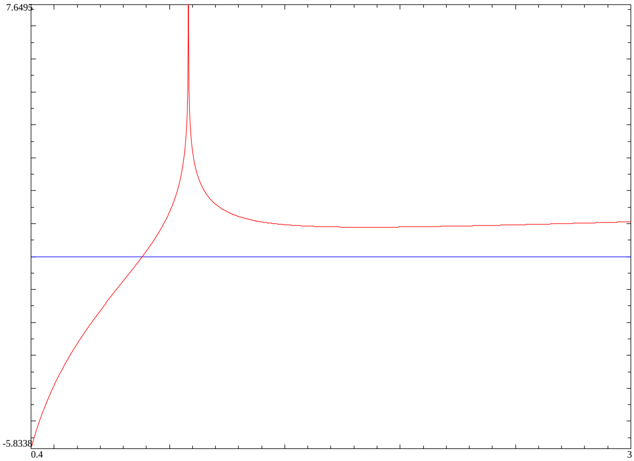
k	ϵ	bound	LMFDB object	accurate
2	1	$N > 10.45$	11a	✓
	-1	$N > 32.95$	37a	
4	1	$N > 4.63$	5.4.1a	✓
	-1	$N > 12.24$	13.4.1a	✓
6	1	$N > 2.85$	3.6.1a	✓
	-1	$N > 6.55$	7.6.1b	✓
8	1	$N > 1.95$	2.8.1a	✓
	-1	$N > 4.09$	5.8.1a	✓
10	1	$N > 1.37$	2.10.1a	✓
	-1	$N > 2.75$	3.10.1b	✓
12	1	$\star$	Δ	
	-1	$N > 1.97$	4.12.1a	

k=12 : Ramanujan Δ function



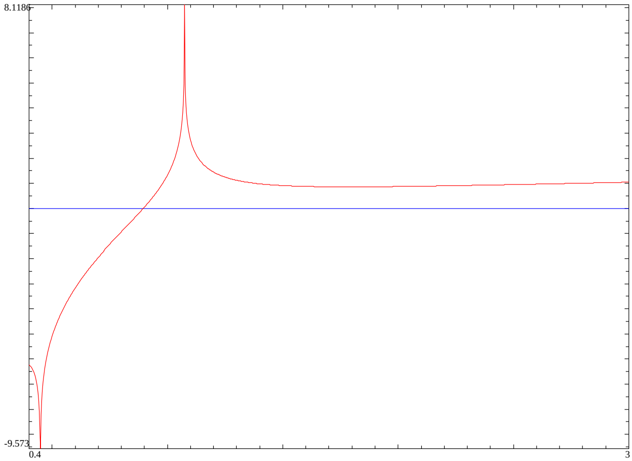
$x = 0.05i, N \geqslant 0.86$

k=12 : Ramanujan Δ function



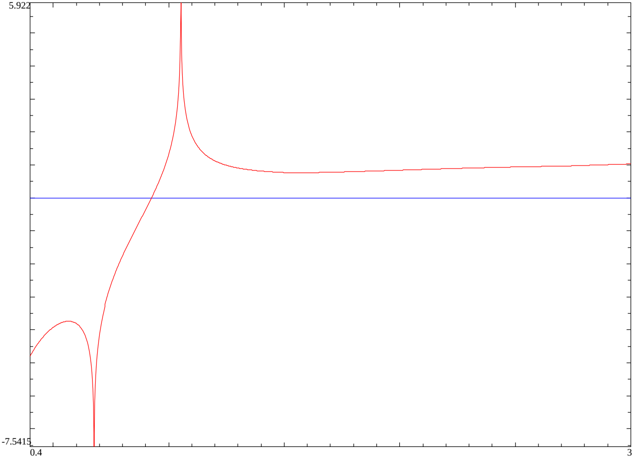
$x = 0.20i$, $N \geqslant 0.88$

k=12 : Ramanujan Δ function



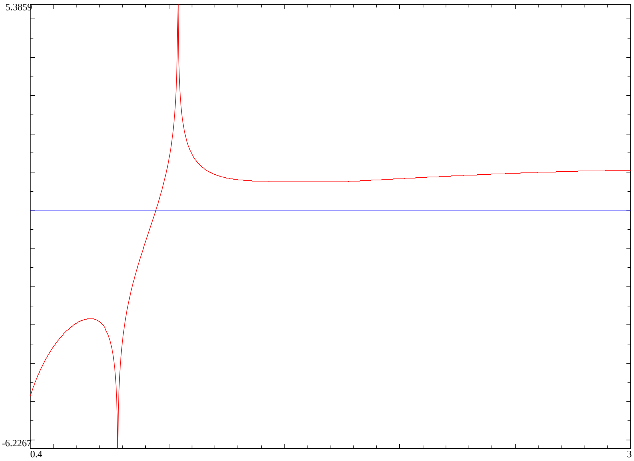
$x = 0.25i,\ N \geqslant 0.90$

k=12 : Ramanujan Δ function



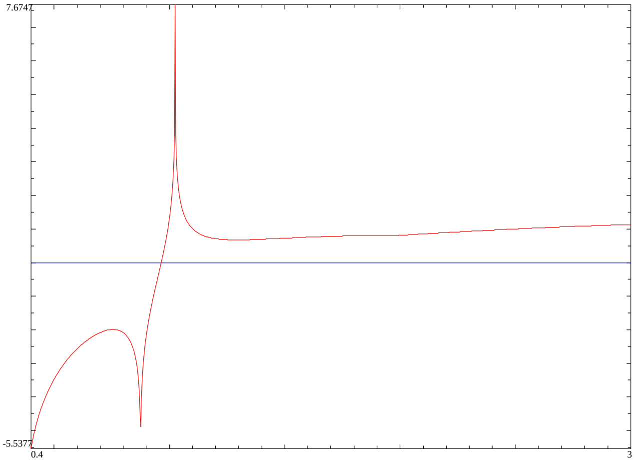
$x = 0.35i$, $N \geqslant 0.93$

k=12 : Ramanujan Δ function



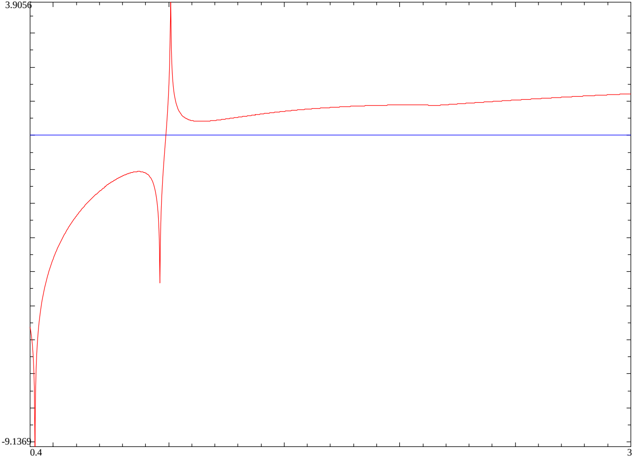
$x = 0.40i, N \geqslant 0.94$

k=12 : Ramanujan Δ function



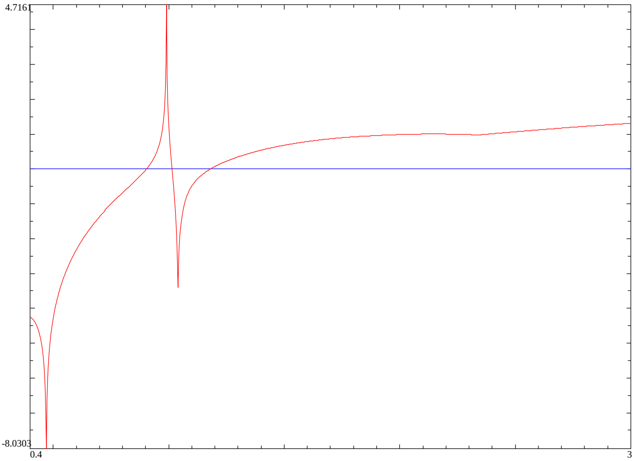
$x = 0.45i, N \geqslant 0.96$

k=12 : Ramanujan Δ function



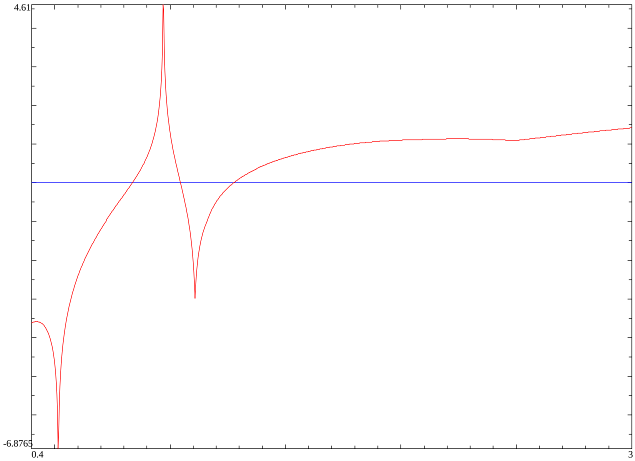
$x = 0.50i, N \geqslant 0.99$

k=12 : Ramanujan Δ function



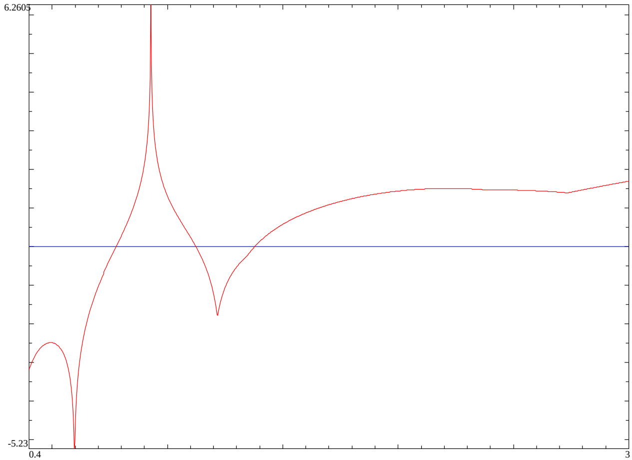
$x = 0.55i$, $0.90 \leqslant N \leqslant 1.01$ or $N \geqslant 1.18$

k=12 : Ramanujan Δ function



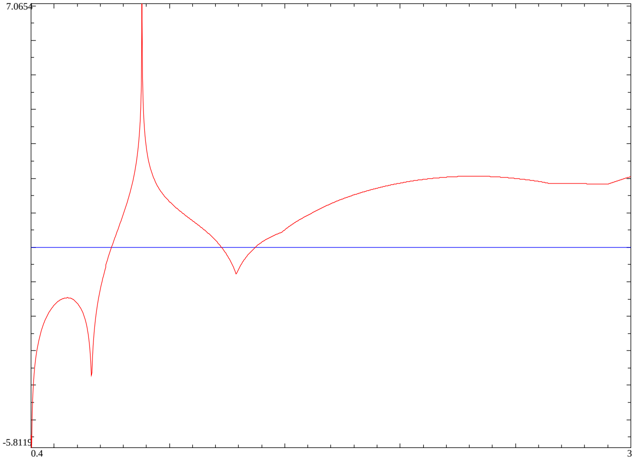
$x = 0.60i$, $0.84 \leqslant N \leqslant 1.04$ or $N \geqslant 1.27$

k=12 : Ramanujan Δ function



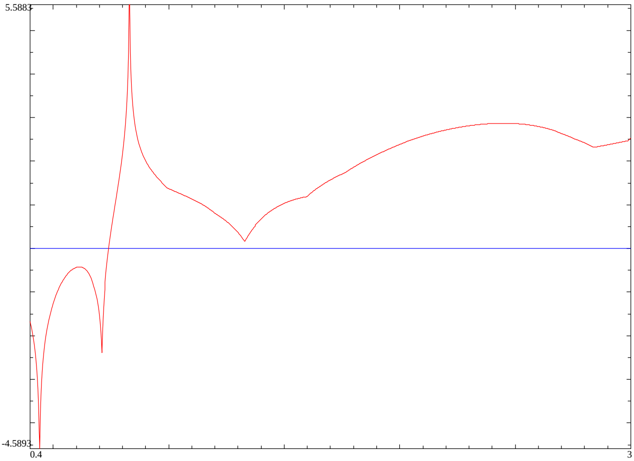
$x = 0.70i$, $0.78 \leqslant N \leqslant 1.12$ or $N \geqslant 1.37$

k=12 : Ramanujan Δ function



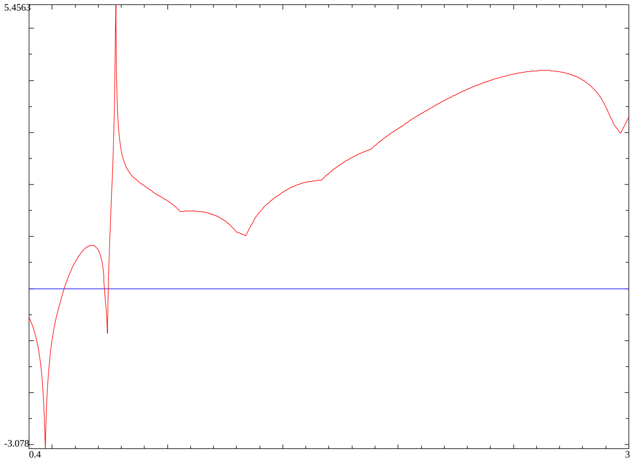
$x = 0.80i, N \geqslant 0.75$

k=12 : Ramanujan Δ function



$x = 0.90i$, $N \geqslant 0.74$

k=12 : Ramanujan Δ function



$x = 1.00i$, $N \geqslant 0.55$

k=12 : Ramanujan Δ function

From an analytic point of vue, it's a miracle that Δ exists. It could not for $N < .9999$, nor $1.00001 < N < 1.37$.

General L functions

- Dirichlet series $L(s) = \sum_{n \geqslant 1} a_n n^{-s}$
- gamma factor of level N and degree d

$$\gamma(s) = N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \lambda_i)$$

- functional equation of weight k

$$\Lambda(s) = L(s)\gamma(s) = \epsilon \overline{\Lambda}(k - s)$$

- Ramanujan bound $a_n \leqslant \sigma_0(n)^{d-1} n^{\frac{k-1}{2}}$
- $+ \ \Lambda$ meromorphic. Here assume holomorphic.

Theta equations

Fourier form

$$\Lambda(s) = \epsilon\overline{\Lambda}(k-s)$$

if and only if for all $x \in \mathbb{R}+]-\frac{d\pi}{4}, \frac{d\pi}{4}[$,

$$F(x) = \epsilon\overline{F}(-x)$$

where

$$F(x) = e^{\frac{k}{2}x}\sum_n a_n \mathscr{M}^{-1}\left[\gamma(s); e^x n\right] \quad (6)$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \Lambda(\frac{k}{2}+it)e^{ixt}\,\mathrm{d}t \quad (7)$$

Inverse Mellin transforms

$\gamma(s)$	$N^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)$	$N^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s)$	$N^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+\nu)$
$\mathscr{M}^{-1}[\gamma; x]$	$e^{-\frac{\pi}{N}x^2}$	$e^{-\frac{2\pi}{\sqrt{N}}x}$	Bessel K_ν

More gamma factors (real shifts) now available in Pari/gp

```
g=gammamellininvinit([0,0,1,1,1])
gammamellininv(g,x)
```

Conclusion For any L function, easy to produce equations $\sum_n a_n x_n$ satisfied by the Dirichlet series, with $x_n \to 0$ exponentially.

Results for weight k=1

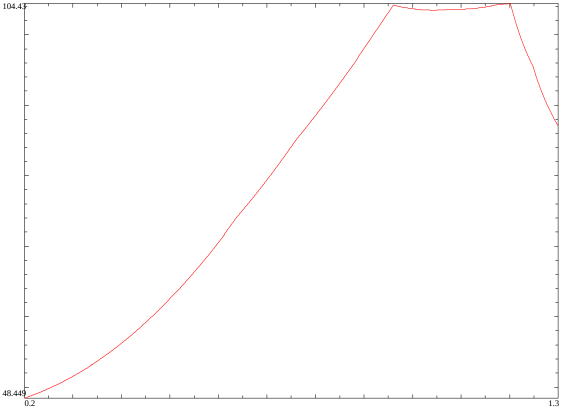
Theorem (Smallest discriminants of number fields)

Let $K/\mathbb{Q}$ be a number field of signature r, s.

- *if* $r, s = 3, 0$, *then* $|\Delta| \geqslant 25$ *;*
- *if* $r, s = 1, 1$, *then* $|\Delta| \geqslant 15$ *;*
- *if* $r, s = 4, 0$, *then* $|\Delta| \geqslant 105$ *;*
- *if* $r, s = 2, 1$, *then* $|\Delta| \geqslant 64$ *;*
- *if* $r, s = 0, 2$, *then* $|\Delta| \geqslant 40$ *;*
- *if* $r, s = 5, 0$, *then* $|\Delta| \geqslant 356$.

"Proof" (case 4,0)

$\zeta_K^*(s) = \zeta_K(s)/\zeta(s)$ is degree 4, weight 1, holomorphic L function, with $\gamma(s) = N^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^4$.



Example : weight k=1

First weight 1 L functions

gamma	N	formulas
[0]	5, 8, 12, ...	$\left(\frac{5}{\cdot}\right), \left(\frac{8}{\cdot}\right), \left(\frac{12}{\cdot}\right), \dots$
[1]	3, 4, 7, ...	$\left(\frac{-3}{\cdot}\right), \left(\frac{-4}{\cdot}\right), \left(\frac{-7}{\cdot}\right), \dots$
[0, 0]	25, 40, 49, ...	$\left(\frac{5^2}{\cdot}\right), \left(\frac{5\times 8}{\cdot}\right), \zeta^\star_{x^3\text{-}x^2\text{-}2*x+1}, \dots$
[0, 1]	15, 20, 23, ...	$\left(\frac{-3\times 5}{\cdot}\right), \left(\frac{-4\times 5}{\cdot}\right), \zeta^\star_{x^3\text{-}x^2+1}, \dots$
[1, 1]	9, 12, 16, ...	$\left(\frac{(-3)^2}{\cdot}\right), \left(\frac{-3\times -4}{\cdot}\right), \left(\frac{-4\times -4}{\cdot}\right), \dots$
[0, 0, 0]	125, 200, 245, ...	$\left(\frac{5^3}{\cdot}\right), \left(\frac{5^2\times 8}{\cdot}\right), \left(\frac{5}{\cdot}\right)\zeta^\star_{x^3\text{-}x^2\text{-}2*x+1}, \dots$
[0, 0, 1]	75, 100, 115, ...	$\left(\frac{-3\times 5^2}{\cdot}\right), \left(\frac{-4\times 5^2}{\cdot}\right), \left(\frac{5}{\cdot}\right)\zeta^\star_{x^3\text{-}x^2+1}, \dots$
[0, 1, 1]	45, 60, 69, ...	$\left(\frac{(-3)^2\times 5}{\cdot}\right), \left(\frac{-3\times -4\times 5}{\cdot}\right), \left(\frac{-3}{\cdot}\right)\zeta^\star_{x^3\text{-}x^2+1}, \dots$
[1, 1, 1]	27, 36, 48, ...	$\left(\frac{(-3)^3}{\cdot}\right), \left(\frac{(-3)^2\times -4}{\cdot}\right), \left(\frac{-3\times(-4)^2}{\cdot}\right), \dots$

Enumerate Dirichlet series

Recall : from the beginning we use

$$|q| \leqslant \sum_{k \geqslant 2} b_k \left|q^k\right|, \text{ where } |a_k| \leqslant b_k. \tag{8}$$

Could push the game further trying values of a_2 in the left-hand side...

$$\left|q + a_2 q^2\right| \leqslant \sum_{k \geqslant 3} b_k \left|q^k\right|, \text{ where } |a_k| \leqslant b_k. \tag{9}$$

for all values $a_2 \in [-b_2, b_2]$.

Enumerate Dirichlet series

goal find all L-functions having specified invariants

input an approximate functional equation $\sum_n a_n x_n = 0$

hypothesis $a_n \in \mathbb{Z}$ + Ramanujan bounds $|a_n| \leqslant b_n$.

1. $a_1 = 1$
2. $a_2 \in [-b_2, b_2]$ s.t.

   $$|(a_1 x_1) + a_2 x_2| \leqslant B_2 = b_3 |x_3| + b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| +$$

3. $a_3 \in [-b_3, b_3]$ s.t.

   $$|(a_1 x_1 + a_2 x_2) + a_3 x_3| \leqslant B_3 = b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| + .$$

4. ...

Enumerate Dirichlet series

goal find all L-functions having specified invariants

input an approximate functional equation $\sum_n a_n x_n = 0$

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3. $a_3 \in [-b_3, b_3]$ s.t.

   $|(a_1 x_1 + a_2 x_2) + a_3(x_3 + a_2 x_6)| \leqslant B_3 = b_4 |x_4| + b_5 |x_5| + \cdots + b_7 |x$

4. ...

More structure on Dirichlet series

- Euler product $L(s) = \prod F_p(p^{-s})^{-1}$
- Euler factor $F_p(T) = 1 + c_{p,1}T + \dots T^d$
- local functional equation

$$c_{p,d-j} = \chi(p) p^{\frac{w}{2}(d-2j)} c_{p,j}$$

  with χ Dirichlet character modulo N

- Ramanujan bounds : using $|\text{roots}| \leqslant p^{\frac{w}{2}}$

$$c_{p,j} \leqslant \binom{d}{j} p^{j\frac{w}{2}}$$

$$a_k \leqslant k^{\frac{w}{2}} \prod_{p^e \| k} \binom{e+d-1}{d-1}$$

Search

Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients $c_{p,e}$ for $e \leqslant \frac{d}{2}$ (or $e \leqslant d-1$ if $p \mid N$).
- Depth-first search (constant in memory)

Examples

```
gp > lfunbuild([[],1,[0,1],2,11,1],45,[2])
time = 11 ms.
%1 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
gp > lfunbuild([[],1,[0,1],2,12,1],45,[2])
time = 11 ms.
%2 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

Search

Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients $c_{p,e}$ for $e \leqslant \frac{d}{2}$ (or $e \leqslant d-1$ if $p \mid N$).
- Depth-first search (constant in memory)

Examples

```
gp> lfunbuild([[],1,[0,1],2,26,1],59,[1])
time = 7 ms.
%3 = [1, 3, 3, 3, 2, 26, 4, 15, 5, 2, 2, 2, 2, 12, 2, 2, 2]
```

Search

Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients $c_{p,e}$ for $e \leqslant \frac{d}{2}$ (or $e \leqslant d-1$ if $p \mid N$).
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Examples

```
gp> lfunbuild([[],1,[0,1],2,26,1],59,[1])
time = 7 ms.
%3 = [1, 3, 3, 3, 2, 26, 4, 15, 5, 2, 2, 2, 2, 12, 2, 2, 2]
```

the choice of equation is still important !

```
gp> lfunbuild([[],1,[0,1],2,26,1],59,[[.42]])
time = 7 ms.
%4 = [1, 3, 3, 2, 3, 2, 4, 2, 3, 2, 2, 2, 2, 2, 4, 2, 2]
```

Program lfunbuild

Two independant functions

- prepare seach tree
    - compute a family of equations
    - identify search variables, ranges, search levels in the tree
    - craft nice equation for each level
    - compute tails B_p
- prune tree using depth first search (constant memory)
    - start at $p = 2$
    - solve $|\text{polynomial}(a_p)| \leqslant B_p$
    - for each possible value a_p
        - propagate value in Dirichlet series
        - recursively descend next level

```
gp > for(N=10,100,print(N,": ",Vec(lfunbuild([[],1,[0,1],2,N,1],31,[2]))))
```

Framework

Write $k \prec p^e$ if k is p-smooth and $p^e \nmid k$.
Assume $\{a_k\}$ known for $k \prec p^e$. Equation for coefficient a_{p^e} :

$$\left(\sum_{k \prec p^e} a_k x_k\right) + a_{p^e} \left(\sum_{m \prec p} a_m x_{p^e m}\right) = \sum_{k \succ p^e} a_k x_k$$

Ramanujan bound $|a_n| \leqslant b_n$ on the tail
$\rightsquigarrow$ polynomial equation $|P(a_{p^e})| \leqslant r$
$\rightsquigarrow$ solve in integers in $[-b_{p^e}, b_{p^e}]$.

Framework

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If $e \geqslant \frac{d}{2}$, by reciprocity of $F_p(T)$ all a_{p^ℓ} in terms of a_{p^j}, $j \leqslant e$

$$\left(\sum_{n \prec p^e} a_n x_n\right) + \sum_{\ell \geqslant e} a_{p^\ell}\left(\sum_{m \prec p} a_m x_{p^\ell m}\right) = \sum_{n \succ p^\infty} a_n x_n$$

Ramanujan bound $|a_n| \leqslant b_n$ on the tail
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$\rightsquigarrow$ solve in integers in $[-b_{p^e}, b_{p^e}]$.

Good and bad tails

Case of polynomial equation of degree 1 :

$$|S_0 + a_p S_1| \leqslant B_p = b_{p_+} |x_{p_+}| + b_{p_{++}} |x_{p_{++}}| + \dots$$

- $|S_1| \geqslant B_p \Rightarrow$ at most one solution a_p.
- the ratio $\frac{B_p}{x_p}$ can be studied a priori
- usually nice at big prime gaps
- can be horrible for twin primes

Combine equations

Combine approximate functional equations :

$$X_n = (x_{n,1}, \dots x_{n,r})$$

define $S = \sum_{n \prec p^e} a_n X_n$, $T = \sum_{m \prec p} a_m X_{p^e m}$, and $R = (b_n X_n)_{n \succ p}$.
For all $\lambda \in \mathbb{R}^n$,

$$|S.\lambda + a_p T.\lambda| \leqslant \|R.\lambda\|_1$$

Choose λ to minimize $\frac{\| R.\lambda\|}{| W.\lambda|}$: "least absolute deviations".
Can be solved with iterated + weighted least squares.

Observations

- Ramanujan Δ very easy to build (despite $k = 12$).

```
gp > lfunbuild([[],1,[0,1],12,1,1],100,2)
time = 42 ms.
%1 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
```

Observations

- Ramanujan Δ very easy to build (despite $k = 12$).
- Same for higher weight, conductor 1

```
gp > lfunbuild([[],1,[0,1],20,1,1],20,[2],1)
time = 20 ms.
%4 = [[1, 456, 50652, -316352, -2377410, 23097312, -16917544, -383331840, 14033
```

Observations

- Ramanujan Δ very easy to build (despite $k = 12$).
- Same for higher weight, conductor 1

```
gp > lfunbuild([[],1,[0,1],20,1,1],20,[2],1)
time = 20 ms.
%4 = [[1, 456, 50652, -316352, -2377410, 23097312, -16917544, -383331840, 14033

gp > mfcoefs(mfsearch([1,20])[1][2],30)
%5 = [0, 1, 456, 50652, -316352, -2377410, 23097312, -16917544, -383331840, 140
```

- For $N = 66$, $k = 2$ and central character $\chi = \left(\frac{-66}{\cdot}\right)$, a match exists for the 71 first primes, then disapears.

Observations

- Ramanujan Δ very easy to build (despite $k = 12$).
- May need many equations to cancel tail (here $11a^{\otimes}2$)
- For $N = 66$, $k = 2$ and central character $\chi = \left(\frac{-66}{\cdot}\right)$, a match exists for the 71 first primes, then disapears. $\left(\frac{-66}{\cdot}\right)$ is trivial up to $p = 19$, and $\pi(19^2) = 72$

Observations

- Ramanujan Δ very easy to build (despite $k = 12$).

- May need many equations to cancel tail (here $11a^{\otimes}2$)
- For $N = 66$, $k = 2$ and central character $\chi = (\frac{-66}{\cdot})$, a match exists for the 71 first primes, then disapears.
- Modular forms expansions 805b and 805c start to differ at primes 11 and 13, with values exchanged

Conclusion

ToDo

- work still in progress... [bugs, precision]
- optimize equations
- try other sources of equations
- write tree search in ball arithmetic (Arb)

Goals

- compute interesting examples.
  Challenges : $\Gamma_{\mathbb{C}}(s-6)\Gamma_{\mathbb{C}}(s)$, $k = 20$, $N = 1$, $a_2 = 0$...
- prove nothing exists outside what is expected
- fill missing bad Euler factors rigorously (e.g. symmetric powers)