Computations with Dedekind-eta Functions and Modular Forms

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Basic Definitions

Dedekind-eta Function

$\eta(z)$ function is the holomorphic function defined on the upper half plane by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty}(1 - e^{2\pi inz}).$$

Theta series and modular forms

Theorem

Let a be a positive integer, and let N be a positive integer such that $N \equiv -1(4a)$. Then the theta series

$$\Theta_a^N(\tau) := \sum_{x,y\in\mathbb{Z}} q^{ax^2+xy+\frac{N+1}{4a}y^2}$$

is an element of $M_1(N,\chi_N)$ where χ_N is the Kronecker character $(\frac{-N}{\cdot})$

Motivation

For $a = 1, 2, 3, 4, 6$, the theta series $\Theta_a^N(\tau)$ can be expressed as a linear combination of some eta-quotients.

Theorem (Ogasawara 2018)

$$\Theta_1^3(\tau) = \frac{\eta(2\tau)^5\eta(6\tau)^5}{\eta(\tau)^2\eta(4\tau)^2\eta(3\tau)^2\eta(12\tau)^2} + 4\frac{\eta(4\tau)^2\eta(12\tau)^2}{\eta(2\tau)\eta(6\tau)}$$

Pari/GP Implementation

```
?A=mffrometaquo([2,5;6,5;1,-2;4,-2;3,-2;12,-2]);
? B=mffrometaquo([4,2;12,2;2,-1;6,-1]);
? mflinear([A,B],[1,4]);
? C=mflinear([A,B],[1,4]);
? Ser(mfcoefs(C,25),q)
%5 = 1 + 6*q + 6*q^3 + 6*q^4 + 12*q^7 + 6*q^9 + 6*q^12 + 12*q^13 + 6*q^16 + 12*q^19 + 12*q^21 +
? F=mfinit([3,1,-3]);
? mfcoefs(F,10)
%7 =
[1/6]

[  1]

[  0]

[  1]

[  1]

[  0]

[  0]

[  2]

[  0]

[  1]
```

Theorem (Ogasawara 2018)

$$\Theta_2^7(\tau) = \Theta_1^7(2\tau) + 2\frac{\eta(2\tau)^2\eta(14\tau)^2}{\eta(\tau)\eta(7\tau)}$$

Pari/GP Implementation

```
? A1=mffrometaquo([4,5;28,5;2,-2;8,-2;14,-2;56,-2]);
? B1=mffrometaquo([8,2;56,2;4,-1;28,-1]);
? C1=mflinear([A1,B1],[1,4]);
? D=mffrometaquo([2,2;14,2;1,-1;7,-1]);
? E=mflinear([C1,D],[1,2]);
? Ser(mfcoefs(E,25),q)
%14 = 1 + 2*q + 4*q^2 + 6*q^4 + 2*q^7 + 8*q^8 + 2*q^9 + 4*q^11 + 4*q^14 + 10*q^16 + 4*q^18 + 8*c
? F1=mfinit([7,1,-7]);
? mfcoefs(F1,10)
%16 =
[1/2]

[  1]

[  2]

[  0]

[  3]

[  0]

[  0]

[  1]

[  4]
```

Theorem (Ogasawara 2018)

$$\Theta_3^{11}(\tau) = \Theta_1^{11}(\tau) + 2\frac{\eta(4\tau)\eta(6\tau)^2\eta(44\tau)\eta(66\tau)^2}{\eta(2\tau)\eta(12\tau)\eta(22\tau)\eta(132\tau)} + 2\frac{\eta(2\tau)^2\eta(3\tau)\eta(12\tau)\eta(22\tau)^2\eta(33\tau)\eta(132\tau)}{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(11\tau)\eta(44\tau)\eta(66\tau)}$$

Pari/GP Implementation

```
? A2=mffrometaquo([6,5;66,5;3,-2;12,-2;33,-2;132,-2]);
? B2=mffrometaquo([12,2;132,2;6,-1;66,-1]);
? C2=mflinear([A2,B2],[1,4]);
? D1=mffrometaquo([4,1;6,2;44,1;66,2;2,-1;12,-1;22,-1;132,-1]);
? E1=mffrometaquo([2,2;3,1;12,1;22,2;33,1;132,1;1,-1;4,-1;6,-1;11,-1;44,-1;66,-1]);
? F=mflinear([C2,D1,E1],[1,2,2]);
? Ser(mfcoefs(F,25),q)
%26 = 1 + 2*q + 4*q^3 + 2*q^4 + 4*q^5 + 6*q^9 + 2*q^11 + 4*q^12 + 8*q^15 + 2*q^16 + 4*q^20 + 4*
? F2=mfinit([11,1,-11]);
? mfcoefs(F2,10)
%28 =
[1/2]

[  1]

[  0]

[  2]

[  1]

[  2]

[  0]

[  0]

[  0]
```

Eisenstein Series

Definition

We define an Eisenstein series E_{t_1,t_2} by

$$E_{t_1,t_2} = C_{t_1,t_2} + \sum_{n=1}^{\infty} \sigma_{\chi_{t_1},\chi_{t_2}}(n)q^n$$

for some constants C_{t_1,t_2} and some characters χ_{t_1}, χ_{t_2}

We define the eta quotients

$$B_1(q) := \frac{\eta(z)\eta^4(6z)\eta^2(8z)}{\eta(2z)\eta(3z)\eta(13z)},\ B_2(q) := \frac{\eta^2(z)\eta^4(8z)\eta^4(12z)}{\eta(4z)\eta(6z)\eta(24z)}.$$

We define the eta quotients

$$C_1(q) := \frac{\eta(z)\eta(4z)\eta^4(6z)\eta^2(24z)}{\eta(2z)\eta(3z)\eta^2(12z)},$$
$$C_2(q) := \frac{\eta^2(z)\eta^4(4z)\eta(6z)\eta(24z)}{\eta^2(2z)\eta(8z)\eta(12z)}.$$

Theorem (Alaca et al, 2017)

Theorem

Assume the setup above. Then $\{E_{1,8}(z), E_{1,8}(3z), E_{8,1}(z), B_1(q), B_2(q)\}$ *is basis for* $M_2(\Gamma_0(24), \chi_8)$.

Pari/GP Implementation

```
? E18=mfeisenstein(2,1,8);
? E183=mfbd(E18,3);
? E81=mfeisenstein(2,8,1);
? E813=mfbd(E81,3);
? B1=mffrometaquo([1,1;6,4;8,2;2,-1;3,-1;12,-1]);
? B2=mffrometaquo([1,2;8,1;12,4;4,-1;6,-1;24,-1]);
? M2=mfinit([24,2,8]);
? mftobasis(M2,E18)
%36 = [1, 0, 0, 0, 0, 0]~
? mftobasis(M2,E183)
%37 = [0, 1, 0, 0, 0, 0]~
? mftobasis(M2,E81)
%38 = [0, 0, 1, 0, 0, 0]~
? mftobasis(M2,E813)
%39 = [0, 0, 0, 1, 0, 0]~
? mftobasis(M2,B1)
%40 = [0, 0, 0, 0, 1/2, 0]~
? mftobasis(M2,B2)
%41 = [0, 0, 0, 0, 1, 1/2]~
```

Theorem (Alaca et al, 2017)

Theorem

Assume the setup above. Then $\{E_{1,12}(z), E_{1,12}(2z), E_{12,1}(z), E_{12,1}(2z), E_{-4,-3}(z), E_{-4,-3}(2z), E_{-3,-4}(z), E_{-3,-4}(2z)\}$ *is basis for* $M_2(\Gamma_0(24), \chi_1 2)$.

Pari/GP Implementation

```
? E112=mfeisenstein(2,1,12);
? E1122=mfbd(E112,2);
? E121=mfeisenstein(2,12,1);
? E1212=mfbd(E121,2);
? E34=mfeisenstein(2,-3,-4);
? E342=mfbd(E34,2);
? E43=mfeisenstein(2,-4,-3);
? E432=mfbd(E43,2);
? M2=mfinit([24,2,12]);
? mftobasis(M2,E112)
%51 = [1, 0, 0, 0, 0, 0, 0, 0]~
? mftobasis(M2,E1122)
%52 = [0, 1, 0, 0, 0, 0, 0, 0]~
? mftobasis(M2,E121)
%53 = [0, 0, 1, 0, 0, 0, 0, 0]~
? mftobasis(M2,E1212)
%54 = [0, 0, 0, 1, 0, 0, 0, 0]~
? mftobasis(M2,E34)
%55 = [0, 0, 0, 0, 1, 0, 0, 0]~
? mftobasis(M2,E342)
%56 = [0, 0, 0, 0, 0, 1, 0, 0]~
? mftobasis(M2,E43)
%57 = [0, 0, 0, 0, 0, 0, 1, 0]~
? mftobasis(M2,E432)
%58 = [0, 0, 0, 0, 0, 0, 0, 1]~
```

Open Problem and Discussion

- Since we have already had eta-quotients for half-integral modular forms then it is natural to ask similar problem.
- Pari/GP is efficient for computations.
- We are open for all suggestions and collaborations.
- This will be PhD thesis problem for Banu Irez Aydin.

References-I

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- T. Ogasawara, Some expressions for binary theta series by eta-quotients and their applications, to appear in Journal of Number Theory, 2018.

Thank you for your attention

Merci á tous!