Biextensions and the Birch-Swinnerton-Dyer conjecture for Calabi-Yau motives

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Quick overview

- Evidence in favor of the BSD conjecture for certain regulators of CY motives
- Get regulators by evaluating solutions of differential equations
- Pass from pure motivic variations to mixed variations by convoluting with the Ur-object

Broader program: CY arithmetic

- Counterparts of known statements for elliptic curves in the world of rank 4 CY motives?
- Modularity: correspondences with Siegel threefolds; associating paramodular forms to known motives
- Central L–value and Deligne's conjecture
- Congruence properties of L–functions
- Central Bloch–Kato; start with analytic rank 0 cases
- Torsion
- Height pairing on CH^2 for a class of rank 4 CY motives whose L-function vanishes to first order at the central argument, and the Birch–Swinnerton-Dyer–type conjecture
- Mirror symmetry and Fano search: connecting arihmetic to the Fano fourfold geometry

Goal

For $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$ Calabi–Yau motives

- Write down regulators for $CH^2(\cdot, 0)$, $CH^3(\cdot, 2)$, $CH^4(\cdot, 4)$
- Connect to $L'(2)$, $L'(1)$, $L''(0)$

Example: $L'(\cdot, 1)$ of an elliptic curve

Let E be the elliptic curve given by $y^2 = x^3 - x + 1/4$.

- Let $P_0 = [0, 1/2]$, and let z_0 be its representative in the fundamental parallelogram.
- Let
$$L(E, s) = \prod_{p \text{ bad}} \prod_{p \text{ good}} (1 - a_p p^{-s} + p \cdot p^{-2s})^{-1}.$$
- Since E is modular, $L(E, s)$ makes sense near $s = 1$ and vanishes at $s = 1$. Numerically,

$L'(E, 1) = 0.305999773834052 \ldots .$

On the other hand,

$$\int_{x_3}^{\infty} \frac{dx}{y} = 2.9934586462319\ldots$$

and

$$\lim_{n\to\infty} \frac{\log \operatorname{denom} (nP)_x}{n^2} = 0.05111140823\ldots$$

We observe

$$L'(E,1) = 2 \cdot \int \cdot \lim .$$

Introduce the Weierstrass sigma function

$$\sigma(z) = z \prod_{\omega \in \Lambda^*} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}\right),$$

and the quasiperiods $\eta_1 = 2\,\frac{\sigma'(\omega_1)}{\sigma(\omega_1)}$, $\eta_2 = 2\,\frac{\sigma'(\omega_2)}{\sigma(\omega_2)}$.

Let p_1, p_2 be the coordinates with respect to the fundamental parallelogram so that $z = p_1\omega_1 + p_2\omega_2$. Put

$$g(z) = -2\log(\sigma(z)) + z(p_1\eta_1 + p_2\eta_2).$$

◀ **Fact.**

$$\operatorname{Re} g(z_0) = 0.05111140823\ldots.$$

What part of it survives for (1,1,1,1) Calabi–Yau motives?

◀ Euler factors take the form

$$\det(1 - T \cdot \mathrm{Frob}_p \,|_{H^3_{\acute{e}t}(\bar{X},\mathbb{Q}_l)}) = 1 + \alpha_p T + \beta_p p T^2 + p^3 \alpha_p T^3 + p^6 T^4.$$

◀ the completed L–function

$$\Lambda(s) = \left(\frac{N}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L(M,s),$$

is believed to be entire and satisfy $\Lambda(s) = \pm\Lambda(4-s)$, where N is the conductor.

◀ Analytic continuation enables one to study the leading coefficient of the Taylor series of $L(s)$ to the left of the convergence halfplane

- To be able to prove anything will most probably need automorphy and modular cycles and higher cycles
- Many believe that a weight 3 paramodular newform f_M could be assocated to such a motive M so that $L(f_M, s) = L(M, s)$.
- A paramodular newform is a Hecke–eigen (3,0)–regular form on the Siegel threefold parametrizing $(1, N)$–polarized abelian surfaces
- Strategy at the moment: prove theorems on the arithmetic geometry/Hodge theory side

Why hypergeometric motives?

- L–functions: Dirichlet series of M + the shape of the functional equation are known (Beukers, Cohen, Mellit, Roberts, Rodriguez Villegas) $\longrightarrow$ Taylor expansion of $L(M, s)$ with arbitrary precision (Dokchitser)
- Betti structures are accessible via gamma structures
- motivic extensions and biextensions can be constructed using the motive of the total family of a hypergeometric pencil

Katz's work

N. Katz introduced implicitly the concept of a hypergeometric motivic sheaf in 1990 by analyzing in detail hypergeometric differential equations and l–adic sheaves

◀ Hypergeometric differential operators: binomials in z

$$L_{\alpha,\beta} = \prod_i (D - \alpha_i) - \lambda z \prod_j (D - \beta_j) \qquad (*)$$

(here $D = z\frac{d}{dz}$).

◀ Katz proved that regular singular hypergeometric differential equations with rational indices (and $\lambda \in \bar{\mathbb{Q}}$) are motivic, i.e., arise in a piece of relative cohomology in a pencil of algebraic varieties defined over a number field.

Betti structures are gamma structures

- If one furthermore requires that the sets $\exp(2\pi i\alpha_i)$'s and $\exp(2\pi i\beta_j)$'s are each $\mathrm{Gal}(\mathbb{Q})$–stable and $\lambda \in \mathbb{Q}$, Katz's construction descends to one defined over $\mathbb{Q}$.
- An analogue of the Picard–Katz holds in a parallel theory of hypergeometric l–adic sheaves: a tame h/g sheaf over $\mathbf{G_m}/\overline{\mathbb{Q}}$ whose local inertia acts quasiunipotently can be defined over a cyclotomic field.
- The descent can be shown to be almost unique in the sense that each two such sheaves differ by a twist by a geometrically constant sheaf of rank 1.
- Similarly, one can descend from D–modules to Hodge modules using gamma structures. A theorem on hypergeometric monodromy would in particular say the following:
- Consider one of the 7 (out of 14) MUM pencils with distinct indices α at the south pole. We will put all $\beta_j = 0$ in what follows.

Put

$$\mathbf{\Gamma}_{\alpha,\beta}(s) = \frac{1}{\prod_{i=1}^{4}\Gamma(s-\alpha_i+1)\prod_{i=1}^{4}\Gamma(-s+\beta_i+1)} \quad (s \in \mathbb{C}),$$

and $\mathrm{A}_j = e^{2\pi i \alpha_j}$.

◀ Put

$$S_{\mathrm{A}_j}(z) = \sum_{l=0}^{\infty} \mathbf{\Gamma}(l+\alpha_j)\, z^{l+\alpha_j}.$$

Then the S_{A_j}'s are solutions to $(*)$ with $\lambda = (-1)^4 = 1$ and the monodromy of $(*)$ around 0 is given by

$$M_0(S_{\mathrm{A}_1}(z), \dots, S_{\mathrm{A}_4}(z))^t = (\mathrm{A}_1 S_{\mathrm{A}_1}(z), \dots, \mathrm{A}_4 S_{\mathrm{A}_4}(z))^t.$$

◀ Denote by V_A the respective Vandermonde matrix

$$\mathrm{V}_\mathrm{A} = \begin{pmatrix} 1 & \mathrm{A}_1 & \cdots & \mathrm{A}_1^3 \\ 1 & \mathrm{A}_2 & \cdots & \mathrm{A}_2^3 \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

The global monodromy of $(*)$ in the basis $V_\mathrm{A}^t(S_{\mathrm{A}_1}(\lambda z), \dots, S_{\mathrm{A}_4}(\lambda z))^t$ can be shown to be defined over $\mathbb{Q}$

◀ and thus, by the Picard–Katz theorem, to underlie a variation $\mathbb{Q}$–Hodge structure of geometric origin.

◀ Define the Hodge filtration as follows: consider the matrix $\Pi_\mathrm{A}(z)$ whose j–th column is $(z\frac{d}{dz})^j V_\mathrm{A}^t(S_{\mathrm{A}_1}(\lambda z), \dots, S_{\mathrm{A}_4}(\lambda z))^t$, and let Fil^{-2-j} be the span of rows $0, \dots, j$ in $\mathbb{Q}^4 \otimes \mathbb{C}$.

Deligne's conjecture

- One expects

$$\frac{L(M_{z_0}, 2)}{\det(2\pi i)^4 \, \mathrm{Re}\, \Pi_{\mathrm{A}}(z_0)_{\{0,1\},\{0,1\}}} \in \mathbb{Q}.$$

- Apparently works.
- Yang 2021, earlier unpublished work by Mellit–G., Roberts, Candelas–de la Ossa–van Straten.

The quadratic twist:

1	$[\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}]$	$[0,0,0,0]$
2	$[\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}]$	$[0,0,0,0]$
3	$[\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}]$	$[0,0,0,0]$
4	$[\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}]$	$[0,0,0,0]$
5	$[\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}]$	$[0,0,0,0]$
6	$[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}]$	$[0,0,0,0]$
7	$[\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}]$	$[0,0,0,0]$.

$\rightsquigarrow$

$\widetilde{1}$	$[-\frac{5}{12}, -\frac{1}{12}, \frac{1}{12}, \frac{5}{12}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
$\widetilde{2}$	$[-\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{2}{5}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
$\widetilde{3}$	$[-\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
$\widetilde{4}$	$[-\frac{1}{3}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{3}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
$\widetilde{5}$	$[-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
$\widetilde{6}$	$[-\frac{3}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{3}{10}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
$\widetilde{7}$	$[-\frac{1}{4}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{4}]$	$[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$.

Twisted families

- Modify $\mathbf{\Gamma}$ accordingly: $\tilde{\mathbf{\Gamma}}_{\tilde{\alpha},\tilde{\beta}}(s) = \tilde{\lambda}^{1/2}\mathbf{\Gamma}_{\tilde{\alpha},\tilde{\beta}}(s)$.
- Twist is expected to raise the 'average' analytic rank in the family, so we need a definition of the biextension VMHS in terms of hypergeometric series
- Theorem: with all the assumptions made, the differential equation

$$DL_{\tilde{\alpha},\tilde{\beta}}\, DS(z) = 0$$

is motivic, i.e. underlies a VMHS of geometric origin.

In addition to the 4 pure periods

$$(\Phi_1(z), \Phi_2(z), \Phi_3(z), \Phi_4(z)) = (S_{\tilde{\mathrm{A}}_1}(\tilde{\lambda} z), \dots, S_{\tilde{\mathrm{A}}_4}(\tilde{\lambda} z)) V_{\tilde{\mathrm{A}}}$$

◀ one introduces an extension solution $\Phi_0(z) := S_1(\tilde{\lambda} z) = \sum_{n=0}^{\infty} \tilde{\boldsymbol{\Gamma}}_{\tilde{\alpha},\tilde{\beta}}(n)(\tilde{\lambda} z)^n$ so that $DL_{\tilde{\alpha},\tilde{\beta}} S_1(\tilde{\lambda} z) = 0 \dots$

◀ …and the (transposed) biextension period matrix

$$\Pi^{\mathrm{biext}}_{\tilde{\mathrm{A}}}(z) = \left((z\frac{d}{dz})^{-1}, 1, \dots, (z\frac{d}{dz})^4\right)^t (\Phi_0(z), \Phi_1(z), \Phi_2(z), \Phi_3(z), \Phi_4(z), 0)$$

◀ The choice of the constant terms in the 0th row is

$$\left((1/\tilde{\alpha}_1 + 1/\tilde{\alpha}_2)\, \tilde{\boldsymbol{\Gamma}}_{\tilde{\alpha},\tilde{\beta}}(0), 0, 0, 0, 0, (2\pi i)\, \tilde{\boldsymbol{\Gamma}}_{\tilde{\alpha},\tilde{\beta}}(0) \right).$$

Birch–Swinnerton-Dyer

- BSD (Bloch, Hain, Kontsevich–Zagier, Scholl):

$$r(z_0) := \frac{L'(M_{z_0}, 2)}{\tilde{\mathbf{\Gamma}}_{\tilde{\alpha},\tilde{\beta}}(0)^{-1}(2\pi i)^8 \det \mathrm{Re}\, \Pi^{\mathsf{biext}}_{\tilde{\mathrm{A}}}(z_0)_{\{0,1,2\},\{0,1,2\}}} \in \mathbb{Q}.$$

- Seems to work for $z_0 = 1/N$:

Examples

α's	t	$r(t)$	conj. value of $t^{-6}\, r(t)$
$[-5/12, -1/12, 1/12, 5/12]$	$1/8$	0.0000065104167	$128/75$
$[-2/5, -1/5, 1/5, 2/5]$	$1/8$	0.00031250000	$2048/25$
$[-2/5, -1/5, 1/5, 2/5]$	$1/3$	0.0070233196	$128/25$
$[-3/8, -1/8, 1/8, 3/8]$	$1/8$	0.000027126736	$64/9$
$[-3/8, -1/8, 1/8, 3/8]$	$1/6$	0.00060966316	$256/9$
$[-3/8, -1/8, 1/8, 3/8]$	$1/2$	0.0069444444	$4/9$
$[-1/3, -1/4, 1/4, 1/3]$	$1/6$	0.014631916	$2048/3$
$[-1/3, -1/4, 1/4, 1/3]$	$1/3$	0.058527664	$128/3$
$[-1/3, -1/4, 1/4, 1/3]$	$1/3$	0.058527664	$128/3$
$[-1/3, -1/4, 1/4, 1/3]$	$1/2$	0.33333333	$64/3$
$[-1/3, -1/6, 1/6, 1/3]$	$1/8$	0.00021701389	$512/9$

Code by Kilian Bönisch

```
? Gamma(alphas,betas,s) = 1/prod(i=1,4,gamma(s-alphas[i]+1)*gamma(-s+betas[i]+1));

checkBSD(alphas,betas,z0) = {
        local(lambda,ord,indicials,f,Log=varlower("Log",z),vandermonde,rescaling,det,M,L);

        lambda = bestappr(exp(sum(i=1,4,psi(1-frac(-alphas[i]))-psi(1-frac(-betas[i])))));
        ord = floor(-100/log(abs(lambda*z0)));

        indicials = concat([0],alphas);
        f = vector(#indicials,i,vector(ord));
        for(i=1,#f,
                f[i][1] = 1;
                for(n=1,ord-1,
                        f[i][n+1] = prod(j=1,#alphas,(n-1+indicials[i]-betas[j])/(n+indicials[i]-alphas[j]))*f[i][n];
                );
                for(n=0,ord-1,if(indicials[i]+n==0,f[i][n+1]*=Log,f[i][n+1]/=indicials[i]+n));
        );
        f = apply(v->Ser(v,z),f);
        indicials = concat(indicials,[0]);
        f = concat(f,[O(z^ord)]);
        f = subst(f,z,lambda*z);
        f = matconcat(vector(6,i,if(i==1,f,vector(6)))~);
        for(i=2,6,for(j=1,6,f[i,j]=z*deriv(f[i-1,j],z)+indicials[j]*f[i-1,j]+deriv(f[i-1,j],Log)));
        f[1,] += [1/alphas[1]+1/alphas[2],0,0,0,0,2*Pi*I];
        vandermonde = matconcat(matdiagonal([1,matrix(4,4,i,j,exp(2*Pi*I*alphas[i]*(j-1))),1]));
        rescaling = matdiagonal(vector(#indicials,i,if(indicials[i]==0,sqrt(lambda),sqrt(lambda))*Gamma(alphas,betas,indicials[i])*lambda^indicials[i]));
        f = substvec(truncate(f),[z,Log],[z0,log(z0)])*matdiagonal(vector(#indicials,i,z0^indicials[i]))*rescaling*vandermonde;
        det = matdet(real(f[1..3,1..3])) / (sqrt(lambda)*Gamma(alphas,betas,0))^3;

        M = hgminit(alphas,betas);
        L = lfunhgm(M,lambda*z0);

        print("alphas=",alphas,", betas=",betas,", t=",z0,": L(M,2)=",lfun(L,2,0)," and r(t)=",lfun(L,2,1)/det);
};


\\ computations

checkBSD([-2/5,-1/5,1/5,2/5],[-1/2,-1/2,-1/2,-1/2],1);
alphas=[-2/5, -1/5, 1/5, 2/5], betas=[-1/2, -1/2, -1/2, -1/2], t=1: L(M,2)=0.E-48 and r(t)=-0.32000000000000000000000000000000000001
```