Stark units and elliptic Gamma functions

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Hilbert's 12th problem

Theorem (Kronecker-Weber)

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Hilbert's 12th problem

Give an explicit description of the abelian extensions of a given field using analytic functions.

Hilbert's 12th problem: imaginary quadratic case

Imaginary quadratic case: Complex multiplication

If $\mathcal{O}_{\mathbb{K}} = [1, \tau]$ then

$$\overline{\mathbb{K}}^{ab} = \bigcup_{m \geq 2} \mathbb{K}(j(\tau), w(1/m, \tau))$$

where j is the j-invariant and w is Weber's function.

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Imaginary quadratic case: Complex multiplication

If $\mathcal{O}_{\mathbb{K}} = [1, \tau]$ then

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Elliptic units (Robert)

Abelian extensions of an imaginary quadratic field $\mathbb{K}$ are given by elliptic units built from Jacobi's θ function:

$$\theta(1/q, \tau)^{12N} / \theta(N/q, N\tau)^{12}$$

for explicit $q, N \in \mathbb{Z}$, $\tau \in \mathbb{K}$.

Rank one abelian Stark conjectures

Consider an abelian extension $\mathbb{L}/\mathbb{K}$ in the following cases:

- $\mathbb{K}$ is totally real and only one infinite place v of $\mathbb{K}$ ramifies in $\mathbb{L}$.
- $\mathbb{K}$ has exactly one complex place v and $\mathbb{L}$ is totally complex.

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Rank one abelian Stark conjectures

Fix a place w of $\mathbb{L}$ above v. There is a unit $u \in \mathbb{L}$ such that for all $\sigma \in Gal(\mathbb{L}/\mathbb{K})$,

$$\zeta'(\sigma, 0) = -\frac{1}{e} \log |u|_w$$

and such that $\mathbb{L}(u^{1/e})/\mathbb{K}$ is abelian.

Totally Real Fields

Theorem (Dasgupta, Kakde, 2023)

Abelian extensions of a totally real field $\mathbb{F}$ are given by the Brumer-Stark units for which there is an explicit analytic p-adic formula, together with the square root function. In other words,

$$\overline{\mathbb{F}}^{ab} = \bigcup_{\substack{\text{Brumer-Stark units} \\ \text{and conjectures}}} \mathbb{F}(u, \sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{n-1}})$$

where $\alpha_1, \ldots, \alpha_r$ are elements of $\mathbb{F}$ representing all possible signs in $\{-1, 1\}^n/(-1, \ldots, -1)$.

Kronecker limit formulae

If $\tau = x + iy$ is a CM point and $Q(u, v) = y^{-1}(u + \tau)(v + \overline{\tau})$ is the associated quadratic form, define the associated ζ function for $\Re(s) > 1$:

$$\zeta_Q(s) = \sum_{m,n \in \mathbb{Z}^2 - \{(0,0)\}} Q(m, n)^{-s}$$

First Kronecker limit formula

$$\lim_{s \to 1} (\zeta_Q(s) - \pi/(s-1)) = 2\pi(\gamma - \log 2 - \log(\sqrt{y}|\eta(\tau)|^2))$$

where η is Dedekind's η function and $\gamma \approx 0.577...$ is Euler's γ constant.

Kronecker limit formulae

If $\tau = x + iy$ is a CM point and $Q(u,v) = y^{-1}(u+\tau)(v+\overline{\tau})$ is the associated quadratic form, define the associated twisted ζ function for $\Re(s) > 1$ and $(u,v) \notin \mathbb{Z}^2$:

$$\zeta_Q(s,u,v) = \sum_{m,n \in \mathbb{Z}^2 - \{(0,0)\}} e^{2i\pi(mu+nv)} Q(m,n)^{-s}$$

$$\theta_1(z,\tau) = \sum_{n \in \mathbb{Z}} \exp(i\pi((n+1/2)^2\tau + (2n+1)(z-1/2)))$$

Second Kronecker limit formula

$$\zeta_Q(1,u,v) = -\pi \log \left| e^{i\pi\tau u^2} \theta_1(v - u\tau, \tau)/\eta(\tau) \right|^2$$

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The theta function

Jacobi's θ function

$$\theta(z,\tau) = \prod_{m\geq 0}(1 - e^{2i\pi(m+1)\tau}e^{-2i\pi z})(1 - e^{2i\pi m\tau}e^{2i\pi z})$$

It is well defined for $\Im(\tau) > 0$ and $z \in \mathbb{C}$. Elliptic units are built from this function and give the Stark units above imaginary quadratic fields.

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Ruijsenaars' elliptic Γ function (1997)

$$\Gamma(z,\tau,\sigma) = \prod_{m,n\geq 0}\frac{1 - e^{2i\pi(m+1)\tau}e^{2i\pi(n+1)\sigma}e^{-2i\pi z}}{1 - e^{2i\pi m\tau}e^{2i\pi n\sigma}e^{2i\pi z}}$$

It is well defined for $\Im(\tau) > 0, \Im(\sigma) > 0$ and $z \notin \mathbb{Z} + \mathbb{Z}_{\leq 0}\tau + \mathbb{Z}_{\leq 0}\sigma$.

Simple properties

Periodicity:

$$\theta(z,\tau) = \theta(z+1,\tau) = \theta(z,\tau+1)$$

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Pseudo-periodicity:

$$\theta(z+\tau,\tau) = -e^{-2i\pi z}\theta(z,\tau)$$

$$\Gamma(z+\tau,\tau,\sigma) = \theta(z,\sigma)\Gamma(z,\tau,\sigma)$$

Modular property for θ

$$\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = \exp(i\pi P_0(z,\tau))\theta(z,\tau)$$

where

$$P_0(z,\tau) = \frac{z^2+z}{\tau} - z + \frac{\tau}{6} + \frac{1}{6\tau} - \frac{1}{2}$$

Modular property for Γ

Theorem (Felder, Varchenko, 1999)

$$\frac{\Gamma\left(\frac{z}{\tau}, \frac{-1}{\tau}, \frac{\sigma}{\tau}\right)}{\Gamma(z,\tau,\sigma)\Gamma\left(\frac{z-\tau}{\sigma}, -\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right)} = \exp(i\pi P_1(z,\tau,\sigma))$$

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where

$$P_1(z,\tau,\sigma) = \frac{z^3}{3\tau\sigma} - \frac{\tau+\sigma-1}{2\tau\sigma}z^2 + \frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{6\tau\sigma}z + \frac{1}{12}(\tau+\sigma-1)\left(\frac{1}{\tau}+\frac{1}{\sigma}-1\right)$$

A Kronecker limit formula for complex cubic fields

Theorem (Bergeron, Charollois, García, 2023)

Let $\mathbb{K}$ be a complex cubic field and $\mathfrak{f} \neq (1)$ be an integral ideal of $\mathbb{K}$. Fix a smoothing ideal $\mathfrak{a}$ of prime norm N coprime to $\mathfrak{f}$. For any integral ideal $\mathfrak{b}$ representing a class in $Cl^+(\mathfrak{f})$, there are explicitly computable $\tau_\mathfrak{b}, \sigma_\mathfrak{b} \in \mathbb{K}$ and an explicitly computable set $F_\mathfrak{b} \subset \mathbb{K}$ such that

$$N\zeta_\mathfrak{f}'([\mathfrak{b}], 0) - \zeta_\mathfrak{f}'([\mathfrak{a}\mathfrak{b}], 0) = \log \left| \prod_{z \in F_\mathfrak{b}} \frac{\Gamma\left(Nz, N\tau_\mathfrak{b}, N\sigma_\mathfrak{b}\right)}{\Gamma\left(z, \tau_\mathfrak{b}, \sigma_\mathfrak{b}\right)^N} \right|^2 = \log |u_\mathfrak{b}|^2$$

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Conjecture (Bergeron, Charollois, García, 2023)

The value $u_\mathfrak{b}$ is the image in $\mathbb{C}$ of a unit inside $\mathbb{K}^+(\mathfrak{f})$ related to the Stark unit.

Multiple elliptic Gamma functions

Nishizawa's G_r functions (2001)

$$G_r(z,\tau_0,\ldots,\tau_r) = \prod_{m_0,\ldots,m_r\geq 0}\left(1-\prod_{j=0}^{r} e^{2i\pi(m_j+1)\tau_j}e^{-2i\pi z}\right) \times\left(1-\prod_{j=0}^{r} e^{2i\pi m_j\tau_j}e^{2i\pi z}\right)^{(-1)^r}$$

These functions are well-defined if $\Im(\tau_j) > 0$ for all $0 \leq j \leq r$ and for $z \notin \mathbb{Z} + \sum_{j=0}^{r}\mathbb{Z}\tau_j$ if r is odd.

We identify $\theta = G_0$ and $\Gamma = G_1$.

Simple properties

Periodicity:

$$G_r(z, \tau_0, \ldots, \tau_r) = G_r(z+1, \tau_0, \ldots, \tau_r) = G_r(z, \tau_0, \ldots, \tau_j + 1, \ldots, \tau_r)$$

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Pseudo-periodicity:

$$G_r(z+\tau_j, \tau_0, \ldots, \tau_r) = G_{r-1}(z, \tau_0, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_r) G_r(z, \tau_0, \ldots, \tau_r)$$

Simple properties

Periodicity:

$$G_r(z, \tau_0, \ldots, \tau_r) = G_r(z+1, \tau_0, \ldots, \tau_r) = G_r(z, \tau_0, \ldots, \tau_j + 1, \ldots, \tau_r)$$

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$$G_r(z+\tau_j, \tau_0, \ldots, \tau_r) = G_{r-1}(z, \tau_0, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_r) G_r(z, \tau_0, \ldots, \tau_r)$$

Example

$$G_2(z+\tau_0, \tau_0, \tau_1, \tau_2) = G_1(z, \tau_1, \tau_2) G_2(z, \tau_0, \tau_1, \tau_2)$$

Bernoulli polynomials

Let $\omega_1, \ldots, \omega_d \in \mathbb{C} - \{0\}$. Then put

$$\sum_{n\geq 0} B^*_{d,n}(z, \omega_1, \ldots, \omega_d)\frac{t^n}{n!} = e^{zt}\prod_{j=1}^{d}\frac{\omega_j t}{e^{\omega_j t} - 1}$$

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$$\sum_{n \geq 0} B^*_{d,n}(z, \omega_1, \ldots, \omega_d) \frac{t^n}{n!} = \left(\sum_{m \geq 0} \frac{z^m t^m}{m!} \right) \prod_{1 \leq j \leq d} \left(\sum_{k_j \geq 0} B_{k_j} \omega_j^{k_j} \frac{t^{k_j}}{k_j!} \right)$$

where $B^*_{d,n}(z, \omega_1, \ldots, \omega_d)$ is a homogeneous polynomial of degree n in $d + 1$ variables with rational coefficients.

Modular property for G_r

Theorem (Narukawa, 2003)

Consider a family $\omega_1, \ldots, \omega_{r+2} \in \mathbb{C} - \{0\}$ such that for all $j \neq n$, $\omega_j/\omega_n \notin \mathbb{R}$. Then the following equality

$$\prod_{j=1}^{r+2} G_r\left(\frac{z}{\omega_j}, \left(\frac{\omega_n}{\omega_j}\right)_{n\neq j}\right) = \exp\left(\frac{-2i\pi}{(r+2)!}\frac{B^*_{r+2,r+2}(z,\omega_1,\ldots,\omega_{r+2})}{\omega_1\omega_2\ldots\omega_{r+2}}\right)$$

holds whenever the left-hand side makes sense as a function of z.

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Geometric construction

- L a rank $r+2$ lattice and $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ its dual space.

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$$\det_B(a_0, \ldots, a_r, \cdot) = s.ev_\gamma(\cdot) \neq 0, \quad s \in \mathbb{Z}_{>0}$$

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Definition

$$G_{r,a_0,\dots,a_r}(w, \sigma, L) = \prod_{\delta \in C^\pm} \left(1 - e^{\pm 2i\pi\left(\frac{\sigma(\delta)-w}{\sigma(\gamma)}\right)}\right)^{(\pm 1)^r}$$

for two explicit cones C^+ and C^- in L.

Remark

There are explicitly computable parameters $\tau_0, \ldots, \tau_r \in \sigma(L) \otimes \mathbb{Q}$ and an explicit finite set $F \subset \mathbb{C}$ such that

$$G_{r,a_0,\ldots,a_r}(w, \sigma, L) = \prod_{z \in F} G_r(z, \tau_0, \ldots, \tau_r)$$

Modular property and equivariance

Theorem (M. 2024)

Let $a_0, \dots, a_{r+1}$ be linear forms in Λ which are linearly independent. There is an explicitly computable Bernoulli polynomial $B_{r+2}(w)$ depending on $a_0, \dots, a_{r+1}$ and σ such that

1. Modular property:

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k \neq j}}(w, \sigma, L)^{(-1)^j} = \exp(i\pi B_{r+2}(w))$$

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1. Modular property:

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k \neq j}}(w, \sigma, L)^{(-1)^j} = \exp(i\pi B_{r+2}(w))$$

2. Equivariance property: $\forall\, g \in \mathrm{SL}_{r+2}(\mathbb{Z})$,

$$G_{r, g\cdot a_0, \ldots, g\cdot a_r}(w, g \cdot \sigma, L) = G_{r, a_0, \ldots, a_r}(w, \sigma, L)$$

Arithmetic construction

- $\mathbb{K}$ an ATR field of degree $d = r + 2 \geq 3$.
- $\mathfrak{f} \neq (1)$ an integral ideal. Set $q\mathbb{Z} = \mathbb{Z} \cap \mathfrak{f}$.
- $\varepsilon_1, \ldots, \varepsilon_r$ fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$.

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- $\mathfrak{a}$ a smoothing ideal of prime norm N.
- $\mathfrak{b}$ integral ideals representing classes in $Cl^+(\mathfrak{f})$.

For $\rho \in \mathfrak{S}_r$, for vectors $h_\rho \in L = \mathfrak{f}\mathfrak{b}^{-1}$ and orientations $\mu_\rho = \pm 1$, define:

$$u_{\rho,j} = \prod_{i=1}^{j} \varepsilon_{\rho(i)}$$

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$$\lambda a_\rho = \mu_\rho \mathrm{det}_B(h_\rho, u_{\rho,1} h_\rho, \ldots, u_{\rho,r} h_\rho, \cdot), \qquad \lambda > 0$$

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$$\lambda a_\rho = \mu_\rho \mathrm{det}_B(h_\rho, u_{\rho,1} h_\rho, \ldots, u_{\rho,r} h_\rho, \cdot), \qquad \lambda > 0$$

$$I_{r,\mathfrak{f},\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \ldots, \varepsilon_r, (h_\rho), (\mu_\rho)) = \prod_{\rho \in \mathfrak{S}_r} G_{r,a_\rho,u_{\rho,1}a_\rho,\ldots,u_{\rho,r}a_\rho}(\sigma(h)/q, \sigma, L)$$

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A conjectural Kronecker limit formula for higher degree

Conjecture

Under some conditions on $\mathfrak{f}$ and $\varepsilon_1, \ldots, \varepsilon_r$, for any integral ideal $\mathfrak{b}$ representing a class in $Cl^+(\mathfrak{f})$, for any permutation $\rho \in \mathfrak{S}_r$, there are explicitly computable vectors $h_{\mathfrak{b},\rho} \in \mathfrak{f}\mathfrak{b}^{-1}$ and signs $\mu_{\mathfrak{b},\rho} \in \{\pm 1\}$ and an explicit finite set $\mathcal{Z}_{\mathfrak{f}}^1$ depending only on $\mathfrak{f}$ such that:

$$u_{\mathfrak{b}} = \prod_{z \in \mathcal{Z}_{\mathfrak{f}}^1} I_{r,\mathfrak{f},\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \ldots, \varepsilon_r, (zh_{\mathfrak{b},\rho}), (\mu_{\mathfrak{b},\rho}))$$

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$$u_{\mathfrak{b}} = \prod_{z \in \mathcal{Z}_{\mathfrak{f}}^1} I_{r,\mathfrak{f},\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \ldots, \varepsilon_r, (z h_{\mathfrak{b},\rho}), (\mu_{\mathfrak{b},\rho}))$$

is the image in $\mathbb{C}$ of a unit inside $\mathbb{K}^+(\mathfrak{f})$ related to the Stark unit and

$$N\zeta_{\mathfrak{f}}'([\mathfrak{b}], 0) - \zeta_{\mathfrak{f}}'([\mathfrak{a}\mathfrak{b}], 0) = \frac{1}{\#\mathcal{Z}_{\mathfrak{f}}^1} \log |u_{\mathfrak{b}}|^2$$

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Computations of the G_r functions

$$G_r(z,\tau_0,\dots,\tau_r) = \begin{cases} \exp\left(\sum_{j\geq 1} \frac{1}{(2i)^r j} \frac{\sin(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is odd} \\ \exp\left(\sum_{j\geq 1} \frac{2}{(2i)^{r+1} j} \frac{\cos(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is even} \end{cases}$$

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This is convergent for $\tau_j \notin \mathbb{R}$ and $|\Im(2z-\sum\tau_j)|<\sum|\Im(\tau_j)|$. If $\Im(\tau_0)<\cdots<\Im(\tau_m)<0<\Im(\tau_{m+1})<\dots,\Im(\tau_r)$ and

$$M=\min\left(-\sum_{j=0}^{m}\Im(\tau_j),\sum_{j=m+1}^{r}\Im(\tau_j)\right)$$

$$\text{Convergence rate:}\quad O\left(\frac{\exp(-2\pi jM)}{j}\right)$$

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This is convergent for $\tau_j \notin \mathbb{R}$ and $|\Im(2z - \sum \tau_j)| < \sum |\Im(\tau_j)|$. If $\Im(\tau_0) < \cdots < \Im(\tau_m) < 0 < \Im(\tau_{m+1}) < \dots, \Im(\tau_r)$ and

$$M = \min\left(-\sum_{j=0}^{m} \Im(\tau_j), \sum_{j=m+1}^{r} \Im(\tau_j)\right)$$

$$\text{Convergence rate: } O\left(\frac{\exp(-2\pi j M)}{j}\right)$$

Problem: in most applications, $M \ll 1$.

```
\p 300
x = exp(log(2)/3)*exp(2*I*Pi/3) \\root of x^3 - 2
t = (x*(x+1) + 3)/15
s = (x - 8)/15
f(z, n, j) = sin(Pi*j*n*(2*z-t-s))/(j*sin(Pi*j*t*n)*sin(Pi*j*s*n))
g(z, n) = suminf(j = 1, n*f(z, 1, j) - f(z, n, j))
exp(-g(-1/3, 5)/(2*I))
algdep(%, 6)
```

Examples

- $X^3 - 14$
- $X^4 - 6X^3 - X^2 - 3X + 1$
- $X^5 - X^4 - X^3 - 2X^2 + X + 1$
- $X^6 - X^5 - 6X^4 - 13X^3 - 8X^2 + 6X + 3$

Challenges: Explicit Cebotarev Density theorem

Theorem (Cebotarev)

The density of degree one primes is uniform in any class group.

Challenges: Explicit Cebotarev Density theorem

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Explicit version 1

Given a class group C, there is a natural number N_C such that for all class in C there is a degree one prime in the class C and whose norm is less than N_C.

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Explicit version 2

Given a class group C and an integer D, there is a natural number $N_{C,D}$ such that for all class in C there is a degree one prime in the class C coprime to D and whose norm is less than $N_{C,D}$.

Challenges: Size of the product

Putting everything together we get:

$$u_{\mathfrak{b}} = \prod_{z \in \mathcal{Z}^1_{\mathfrak{f}}} \prod_{\rho \in \mathfrak{S}_r} \prod_{z \in F_\rho} (\text{Ordinary Gr function})$$

Challenges: Compute F

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- We use **mathnf**.
- If $\det(M) \gg 1$, computation time becomes overwhelming (So does memory usage!)
- As the degree grows, computation time for F increases rapidly.

Challenges: finding the best fundamental units

Everything in the construction depends heavily on the initial choice of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$.

- We can decide if $\varepsilon_1, \ldots, \varepsilon_r$ is better or worse than $\varepsilon'_1, \ldots, \varepsilon'_r$.
- We don't know how to find the best system $\varepsilon_1, \ldots, \varepsilon_r$.

Challenges: position of parameters

$$G_r(z,\tau_0,\dots,\tau_r) = \begin{cases} \exp\left(\sum_{j\geq 1} \frac{1}{(2i)^r j} \frac{\sin(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is odd} \\ \exp\left(\sum_{j\geq 1} \frac{2}{(2i)^{r+1} j} \frac{\cos(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is even} \end{cases}$$

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It is often the case that the convergence rate looks like

$$\frac{0.999999^j}{j}$$

For $j = 2.10^8$ we get 6.10^{-96}. In most cases, this final step in the computations takes 99% of the total computation time.

Challenges: Recognizing the result

- $u_{\mathfrak{b}} \in \mathbb{K}^+(\mathfrak{f})$? Use **algdep** or build the relative polynomial over $\mathbb{K}$ and identify coefficients using **lindep**.
- $N\zeta_{\mathfrak{f}}'([\mathfrak{b}], 0) - \zeta_{\mathfrak{f}}'([\mathfrak{a}\mathfrak{b}], 0) = \frac{1}{\#\mathcal{Z}_{\mathfrak{f}}^1} \log |u_{\mathfrak{b}}|^2$? Use **lindep**.

Thank you!