A matter of definition.

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Abstract

An alternative approach to the arithmetical definition of binomial coefficient. Useful for both purposes, the calculation of $\binom{n}{k}$ and the generation/enumeration of combinations in lexical order.

Contents

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1 Yes!, a matter of definition.

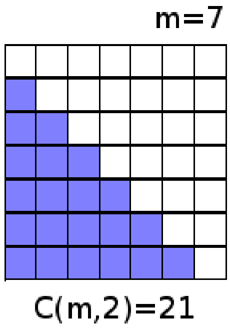
1.1 The standard definition of binomial coefficient in combinatorics

In general, for $n \geq 0$ and $0 \leq k \leq n$; by definition:

$$\mathbb{C}\left(n,k\right) = \frac{1}{k!}\prod_{j=1}^{k}\left(n-k+j\right) \tag{1}$$

And the possible extension for negative integers won't be treated here.

Starting from this definition, and by interpretative observations about simple geometrical constructions made using cubes, constructions like pyramids and another "shapes" for counting its blocks as the numbers of the form $\binom{n}{k}$, we might propose alternative ways of calculation trying to change the standard definition given above.



For example, there in the picture it is strongly suggested the use of a double sum for the computation of binomial coefficients of the form $\binom{n}{2}$. It is clearly not the fastest way to perform such kind of calculations and where the speed is important, the standard definition or expr. (1) is enough. However, one might be also interested in the generation of the so called *binary combinations*, and particularly in the enumeration of combinations in lexical ascending or lexical descending order.

2 Playing a little with cubic blocks and elementary Algebra

2.1 Mathematical expressions are not cubic blocks but fun is allowed there

There are no precise recipes, methods or guidelines to follow when a human is using the imagination and the creativity. Just by inspiration and of course based on the described pyramids and "shapes" the following non-conventional treatment might be found out:

Let imagine the reader a worksession with some *CAS*[1] software with a similar syntax to PARI-GP[2]

$$\mathbb{C}\,(n,k) = \frac{1}{k!}\prod_{j=1}^{k}(n-k+j)\,;$$

$$rhs\,(\star)\,;$$

$$\rightarrow \frac{1}{k!}\prod_{j=1}^{k}(n-k+j)$$

$$subst\left(\star, \frac{1}{k!}, 1\right)\,;$$

$$\rightarrow \prod_{j=1}^{k}(n-k+j)$$

$$subst\left(\star, \prod, \sum\right)\,;$$

$$\rightarrow \sum_{j=1}^{k}(n-k+j)$$

... ...

[1] CAS = "Computer Algebra System"

[2] Where it is used: "rhs(E)" standing for the *right-hand side*, where "E" is some equality expression. Also "subst(P,Q,S)" assumed to be: "In the compound expression P, replace the expression Q by the expression S". It is used a "$\star$" denoting: *the last result* (an expression), and an arrow preceding some expression means that it is a result replied by the system.

… … …

$$\rightarrow \sum_{j=1}^{k} (n-k+j)$$

$$subst\,(\star, -k, +\alpha)\,;$$

$$\rightarrow \sum_{j=1}^{k} (n+\alpha+j)$$

$$subst\,(\star, k, \beta)\,;$$

$$\rightarrow \sum_{j=1}^{\beta} (n+\alpha+j)$$

$$swap\,(\star, \beta, (n+\alpha+j))\,;$$

$$\rightarrow \sum_{j=1}^{(n+\alpha+j)} \beta$$

$$swap\left(\star, \sum_{j=1}^{(n+\alpha+j)}, \beta\right);$$

$$\rightarrow \beta \sum_{j=1}^{(n+\alpha+j)}$$

$$subst\,(\star, j, \psi_j)\,;$$

$$\rightarrow \beta \sum_{\psi_j=1}^{(n+\alpha+j)}$$

$$subst\left(\star, 1, \psi_{(j-1)}+1\right);$$

$$\rightarrow \beta \sum_{\psi_j=\psi_{(j-1)}+1}^{(n+\alpha+j)}$$

$$subst\,(\star, +\alpha, -k)\,;$$

$$\rightarrow \beta \sum_{\psi_j=\psi_{(j-1)}+1}^{(n-k+j)}$$

… … …

... ...

$$\to \beta \sum_{\psi_j=\psi_{(j-1)}+1}^{(n-k+j)}$$

$$subst\left(\star, \beta, \prod_{j=1}^{k}\right);$$

$$\to \prod_{j=1}^{k} \sum_{\psi_j=\psi_{(j-1)}+1}^{(n-k+j)}$$

And it is noteworthy the fact that, at least in this case, all these symbolic transformations made with the aid of an hypothetical *CAS* software, are neither arbitrary nor random. All of this come to be intuitively inspired by geometrical constructions like the picture shown previously in representation of $\binom{7}{2} = 21$.

Well, according to our worksession, it might be possible and correct to state:

$$\mathbb{C}\,(n,k) = \prod_{j=1}^{k} \sum_{\psi_j=\psi_{(j-1)}+1}^{(n-k+j)}$$

But: *Let us wait a moment!*. What is the precise meaning of the right-hand side in such expression?.

In order to answer this question, it might be instructive to re-write our proposal as:

$$\mathbb{C}\,(n,k) = \left\{ \prod_{j=1}^{k} \sum_{\psi_j=\psi_{(j-1)}+1}^{(n-k+j)} \right\} 1 \qquad (2)$$

reading it's right-hand side as an operator applied to the unit.

2.2 Definition.

Going back to the worksession we remember that there is no actually any product, since we replaced $\prod$ with $\sum$. Just for such reason, it might be called (*Why Not?*) a "pseudo-product". Then the so coined "pseudo-product" is actually a multiple summation where the lower indices, the upper indices or both kind of indices are not all independent because there exists a recurrence relation among them.

2.3 Example.

For the particular study case of our current interest: An alternative definition for $\binom{n}{k}$, the lower indices are in monotonic linear recurrence while each upper index is a difference, relative to n and k. Since all what was mentioned is temporarily just an hypothesis, let us try to give justice to the picture in representation of $\binom{7}{2} = 21$ by performing such calculation with a "pseudo-product". By direct substitution in (2), we have (the semicolons are present there for readability):

$$\begin{aligned}
\mathbb{C}(7,2) &= \left\{ \prod_{j=1}^{2} \sum_{\psi_j=\psi_{(j-1)}+1}^{(7-2+j)} \right\} 1 \\
&= \left\{ \prod_{j=1}^{2} \sum_{\psi_j=\psi_{(j-1)}+1}^{(5+j)} \right\} 1 \\
&= \left\{ \sum_{\psi_1=\psi_{(1-1)}+1;}^{(5+1)} \sum_{\psi_2=\psi_{(2-1)}+1}^{(5+2)} \right\} 1 \\
&= \left\{ \sum_{\psi_1=\psi_0+1;}^{6} \sum_{\psi_2=\psi_1+1}^{7} \right\} 1
\end{aligned}$$

And the job hangs up due a subtle detail: ψ_0 is unknown. Since it is not a surprise that correctly stated recurrence relations always include one or more initial conditions, and for simplicity ψ_0 could be defined as an arbitrary constant. Then[3]: $(\psi_0 = 0)$

$$\begin{aligned}
\mathbb{C}(7,2) &= \left\{ \sum_{\psi_1=1;}^{6} \sum_{\psi_2=\psi_1+1}^{7} \right\} 1 \\
&= \left\{ \sum_{\psi_2=1+1}^{7} + \sum_{\psi_2=2+1}^{7} + \sum_{\psi_2=3+1}^{7} + \sum_{\psi_2=4+1}^{7} + \sum_{\psi_2=5+1}^{7} + \sum_{\psi_2=6+1}^{7} \right\} 1 \\
&= \left\{ \sum_{\psi_2=2}^{7} + \sum_{\psi_2=3}^{7} + \sum_{\psi_2=4}^{7} + \sum_{\psi_2=5}^{7} + \sum_{\psi_2=6}^{7} + \sum_{\psi_2=7}^{7} \right\} 1
\end{aligned}$$

$$6 + 5 + 4 + 3 + 2 + 1 = \mathbf{21}$$

[3] The particular structure of "pseudo-product" proposed as replacement for the standard def. of $\binom{n}{k}$ in some sense generalizes the process of counting the number of k-ary combinations that can be made from n elements (The blue blocks in the picture for our example).

2.4 Enumerating combinations in lexical ascending order

Based on the expression (2), a computer program where the summation is replaced with a *for* loop or any similar iterative structure, will be able to print a list in ascending order, consisting in all the possible combinations without repetitions that can be made from n things placing them in groups of k elements each time. All what the program should do is to run the multiple loop printing each time the set of values for the indices ψ_j in the same order that these indices appears in the formula. This same principle might be used for any other kind of enumeration (not only integer numbers) where the idea of k-ary combinations is involved.

2.5 Example:

In *PARI-GP*[4], the following function perform such kind of task:

```
list_nC2(x)=for(psi1=1,x-1,
  print(" ");
  for(psi2=psi1+1,x,
    print1(" (",psi1,",",psi2,");")
  )
);
```

When it is called for $x = 7$, the resulting output is:

```
? list_nC2(7)

 (1,2); (1,3); (1,4); (1,5); (1,6); (1,7);
 (2,3); (2,4); (2,5); (2,6); (2,7);
 (3,4); (3,5); (3,6); (3,7);
 (4,5); (4,6); (4,7);
 (5,6); (5,7);
 (6,7);
```

And there are possible modifications to such code that enables us to change the order for one or both components in these pairs.

[4]Note: The source code shown here was splited for readability, however when it is inserted inside a PARI-GP active session, the user should enter it without the white spaces in order to work.

3 Simplifying the recurrence relations

An inversion applied to both limits in the loops over another way of writing the recurrence relations,

```
rev_list_nC2(x)=forstep(psi1=x,1,-1,
  print("␣");
  forstep(psi2=psi1-1,1,-1,
    print1("␣(",psi1,",",psi2,");")
  )
);
```

produces the following output:

```
? rev_list_nC2(7)

(7,6); (7,5); (7,4); (7,3); (7,2); (7,1);
(6,5); (6,4); (6,3); (6,2); (6,1);
(5,4); (5,3); (5,2); (5,1);
(4,3); (4,2); (4,1);
(3,2); (3,1);
(2,1);
```

Where all the items are shown in lexical descending order. The corresponding summation formula associated to this other sourcecode is:

$$\mathbb{C}(n,k)=\left\{\prod_{j=1}^{k}\sum_{\psi_j=1}^{\psi_{(j-1)}-1}\right\}1 \tag{3}$$

Where this time the proper initial contidition is: $\psi_0 = n+1$.

A full proof for this kind of "pseudo-product" when $k=2$, will be found at the end of the present work.

4 Proof for $k = 2$ of equation 3

The general structure for the identity in the expression 3, was found empirically while solving another problem in Mathematical Physics about certain invariance and autosimilarity present in the Newton's second law after it is generalised to any temporal derivative of the position taken as dependence of the potential energy. Here is what was written originally.

$$(\lambda \geq \mu)$$

$$\Phi(\beta) = \sum_{\mu=0}^{\beta} \sum_{\lambda=0}^{\beta} \mathbf{f}(\mu, \lambda) =$$

$$\left(\sum_{\mu=0}^{\beta} \cdot \sum_{\lambda=0}^{\beta} \right) \mathbf{f}(\mu, \lambda) =$$

$$\left(\sum_{\mu=\lambda=0}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} \right) \mathbf{f}(\mu, \lambda) =$$

$$\left(\sum_{\mu=\lambda=0}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} \right) \mathbf{f}(\mu, \lambda)$$

$$\gamma = (\beta - 1)$$

$$-\Phi_\gamma = -\left(\sum_{\mu=\lambda=0}^{\gamma} + \sum_{\mu=0}^{(\gamma-1)} \sum_{\lambda=\mu+1}^{\gamma} \right) \mathbf{f}(\mu, \lambda)$$

$$\Phi_\beta = \left(\sum_{\mu=\lambda=0}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} \right) \mathbf{f}(\mu, \lambda)$$

$$\Phi_\beta - \Phi_\gamma = \left(\sum_{\mu=\lambda=0}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} \right) \mathbf{f}(\mu, \lambda) - \left(\sum_{\mu=\lambda=0}^{\gamma} + \sum_{\mu=0}^{(\gamma-1)} \sum_{\lambda=\mu+1}^{\gamma} \right) \mathbf{f}(\mu, \lambda)$$

$$\Delta\Phi = \left[\left(\sum_{\mu=\lambda=0}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} \right) - \left(\sum_{\mu=\lambda=0}^{\gamma} + \sum_{\mu=0}^{(\gamma-1)} \sum_{\lambda=\mu+1}^{\gamma} \right) \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=0}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=\lambda=0}^{\gamma} - \sum_{\mu=0}^{(\gamma-1)} \sum_{\lambda=\mu+1}^{\gamma} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=0}^{\beta} - \sum_{\mu=\lambda=0}^{\gamma} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{(\gamma-1)} \sum_{\lambda=\mu+1}^{\gamma} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=0}^{\beta} - \sum_{\mu=\lambda=0}^{[\beta-1]} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{([\beta-1]-1)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \cancel{\sum_{\mu=\lambda=0}^{[\beta-1]} - \sum_{\mu=\lambda=0}^{[\beta-1]}} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{([\beta-1]-1)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{([\beta-1]-1)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$\Delta\Phi = \left[\sum_{\mu=\lambda=\beta}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{([\beta-1]-1)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \left[\sum_{\mu=0}^{(\beta-2)} + \sum_{\mu=\beta-1}^{(\beta-1)} \right] \sum_{\lambda=\mu+1}^{\beta} - \sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\mu+1}^{\beta} + \sum_{\mu=\beta-1}^{(\beta-1)} \sum_{\lambda=\beta}^{\beta} - \sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\mu+1}^{[\beta-1]} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \cancel{\sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\mu+1}^{\beta-1}} + \sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\beta}^{\beta} + \sum_{\mu=\beta-1}^{(\beta-1)} \sum_{\lambda=\beta}^{\beta} - \cancel{\sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\mu+1}^{[\beta-1]}} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \sum_{\mu=0}^{(\beta-2)} \sum_{\lambda=\beta}^{\beta} + \sum_{\mu=\beta-1}^{(\beta-1)} \sum_{\lambda=\beta}^{\beta} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \left[\sum_{\mu=0}^{(\beta-2)} + \sum_{\mu=\beta-1}^{(\beta-1)} \right] \sum_{\lambda=\beta}^{\beta} \right] \mathbf{f}(\mu, \lambda)$$

$$= \left[\sum_{\mu=\lambda=\beta}^{\beta} + \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\beta}^{\beta} \right] \mathbf{f}(\mu, \lambda)$$

$$= \mathbf{f}(\beta, \beta) + \sum_{\mu=0}^{(\beta-1)} \mathbf{f}(\mu, \beta)$$

$$\Delta\Phi(\zeta) = \mathbf{f}(\zeta, \zeta) + \sum_{\mu=0}^{(\zeta-1)} \mathbf{f}(\mu, \zeta)$$

$$\sum_{\zeta=0}^{\beta} \Delta\Phi(\zeta) = \sum_{\zeta=0}^{\beta} \left[\mathbf{f}(\zeta, \zeta) + \sum_{\mu=0}^{(\zeta-1)} \mathbf{f}(\mu, \zeta) \right] = \Phi(\beta)$$

Note:

The following convention was adopted: If $\omega < \alpha$ then $\sum_{\mathbf{i}=\alpha}^{\omega} (\cdots) = \mathbf{0}$; the dots means any expression enclosed by such sum.

$$\Phi(\beta) = \sum_{\zeta=0}^{\beta} \mathbf{f}(\zeta, \zeta) + \sum_{\zeta=0}^{\beta} \sum_{\mu=0}^{(\zeta-1)} \mathbf{f}(\mu, \zeta)$$

From there: When it is defined $\mathbf{f}(\mathbf{u}, \mathbf{w}) = \mathbf{1}$ for every pair $(\mathbf{u}, \mathbf{w})$, it can be stated that:

$$\binom{\beta+1}{2} = \sum_{x=0}^{\beta} \sum_{y=0}^{(x-1)} \mathbf{f}(y, x) = \sum_{\mu=0}^{(\beta-1)} \sum_{\lambda=\mu+1}^{\beta} \mathbf{f}(\mu, \lambda)$$