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1 Systems of coordinates, connections

Let Λ be a Dedekind domain with field of fractions C. Systems of coordinates for a (nonzero) finitely generated projective module P over Λ arise directly from the splitting of a map $\Lambda^g \twoheadrightarrow P$ determining a set of g generators for P. Because Λ is a Dedekind ring the situation simplifies. First, by Steinitz' theorem, P has the form $\Lambda^{r-1} \oplus \mathfrak{a}$ with $\mathfrak{a}$ a nonzero integral ideal of Λ. The case of a free module being straightforward, we are reduced to studying systems of coordinates for $\mathfrak{a}$.

If we wish to consider a set of g generators for $\mathfrak{a}$ (in general, we may take $g \le 2$), then we start with a map $\pi : \Lambda^g \twoheadrightarrow \mathfrak{a}$ with the generators of $\mathfrak{a}$ given by $\pi(e_j)$ where $e_j = (\delta_{ij})_i$ (using the Kronecker delta) represents the standard basis of Λ^g. Since $\mathfrak{a}$ is projective, π admits a splitting $\sigma = (\sigma_1, \dots, \sigma_g) : \mathfrak{a} \to \Lambda^g$ with $\pi \circ \sigma = \mathrm{id}_{\mathfrak{a}}$. Each $\sigma_j \in \mathrm{Hom}_\Lambda(\mathfrak{a}, \Lambda)$.

Lemma 1. Let $\mathfrak{a}^*$ be the fractional ideal inverse to $\mathfrak{a}$; that is, $\mathfrak{a}\mathfrak{a}^* = \Lambda$. Then there is a canonical isomorphism $\mathfrak{a}^* \to \mathrm{Hom}_\Lambda(\mathfrak{a}, \Lambda)$.

Proof. Define $\mu : \mathfrak{a}^* \to \mathrm{Hom}_\Lambda(\mathfrak{a}, \Lambda)$ by sending any $a^* \in \mathfrak{a}^*$ to the map $a \mapsto aa^* \in \Lambda$. In the other direction, because for $a, b \in \mathfrak{a} - \{\mathfrak{o}\}$ and $\varphi \in \mathrm{Hom}_\Lambda(\mathfrak{a}, \Lambda)$ we have

$$\varphi(a)b = \varphi(ab) = a\varphi(b),$$

φ determines a unique element, written as, namely $\nu(\varphi) = \varphi(a)a^{-1} = \varphi(b)b^{-1}$ of C. It is easy to check that the homomorphisms μ and ν are mutually inverse. □

Thus, we can abuse notation and equally write $\sigma : \mathfrak{a} \to \Lambda^g$ as $\sigma = (\sigma_1, \dots, \sigma_g) \in (\mathfrak{a}^*)^g$ where for $a \in \mathfrak{a} - \{0\}$ we have $\sigma_j(a) = \sigma_j a$. It is often convenenient to write $s_j = \pi(e_j)$. Then the fact $\pi \circ \sigma = \mathrm{id}_{\mathfrak{a}}$ corresponds, for each $a \in \mathfrak{a} - \{0\}$, to

$$a = \pi(\sigma_1 a, \dots, \sigma_g a) = \pi\left(\sum (\sigma_j a) e_j\right) = \sum (\sigma_j a)\pi(e_j) = a \sum \sigma_j s_j,$$

from which we deduce that $\sum \sigma_j s_j = 1$. Conversely, given elements $s_1, \dots, s_g \in \mathfrak{a}$ and $\sigma_1, \dots, \sigma_g \in \mathfrak{a}^*$, these displayed equations reveal that when $\sum \sigma_j s_j = 1$ we obtain $a = \sum s_j \sigma_j a$, whence $(s_1, \dots, s_g; \cdot\sigma_1, \dots, \cdot\sigma_g)$ forms a system of coordinates for $\mathfrak{a}$ in the classical sense. Since $\mathfrak{a}^{**} = \mathfrak{a}$, we also see that $(\sigma_1, \dots, \sigma_g; \cdot s_1, \dots, \cdot s_g)$ is equally a system of coordinates for $\mathfrak{a}^*$. In sum, we have the following.

Lemma 2. A system of coordinates for a nonzero ideal $\mathfrak{a}$ of Λ is precisely a pair of g-tuples $(s_1, \dots, s_g) \in \mathfrak{a}^g$ and $(\sigma_1, \dots, \sigma_g) \in (\mathfrak{a}^*)^g$, written $\mathscr{S} = (s_1, \dots, s_g; \sigma_1, \dots, \sigma_g)$, such that $\sum \sigma_j s_j = 1$. Equivalently, such pairs of g-tuples written as $\mathscr{S}^* = (\sigma_1, \dots, \sigma_g; s_1, \dots, s_g)$ are equivalent to systems of coordinates for $\mathfrak{a}^*$. In general, we may choose $g = 1$ when $\mathfrak{a}$ is principal, and $g = 2$ otherwise. Every nonzero ideal of Λ admits systems of coordinates. □