

# Algebraic Number Theory

(PARI-GP version 2.13.3)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ )      `qfb(a, b, c, {d})`  
 reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ )      `qfbred(x, {flag}, {D}, {l}, {s})`  
 return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced      `qfbreds12(x)`  
 composition of forms       $x*y$  or `qfbnucomp(x, y, l)`  
 $n$ -th power of form       $x^n$  or `qfbnupow(x, n)`  
 composition without reduction      `qfbcomprow(x, y)`  
 $n$ -th power without reduction      `qfbpowrow(x, n)`  
 prime form of disc.  $x$  above prime  $p$       `qfbprimeform(x, p)`  
 class number of disc.  $x$       `qfbclassno(x)`  
 Hurwitz class number of disc.  $x$       `qfbhclassno(x)`  
 solve  $Q(x, y) = n$  in integers      `qfbsolve(Q, n)`

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$       `quadgen(x)`  
 minimal polynomial of  $\omega$       `quadpoly(x)`  
 discriminant of  $\mathbf{Q}(\sqrt{x})$       `quaddisc(x)`  
 regulator of real quadratic field      `quadregulator(x)`  
 fundamental unit in real  $\mathbf{Q}(\sqrt{D})$       `quadunit(D, {'w'})`  
 class group of  $\mathbf{Q}(\sqrt{D})$       `quadclassunit(D, {flag}, {t})`  
 Hilbert class field of  $\mathbf{Q}(\sqrt{D})$       `quadhilbert(D, {flag})`  
 ... using specific class invariant ( $D < 0$ )      `polclass(D, {inv})`  
 ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$       `quadray(D, f, {flag})`

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ . We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A  $nf$  structure contains a maximal order and allows operations on elements and ideals. A  $bnf$  adds class group and units. A  $bnr$  is attached to ray class groups and class field theory. A  $rmf$  is attached to relative extensions  $L/K$ .

init number field structure  $nf$       `nfinit(f, {flag})`  
 known integer basis  $B$       `nfinit([f, B])`  
 order maximal at  $vp = [p_1, \dots, p_k]$       `nfinit([f, vp])`  
 order maximal at all  $p \leq P$       `nfinit([f, P])`  
 certify maximal order      `nfcertify(nf)`

### nf members:

a monic  $F \in \mathbf{Z}[X]$  defining  $K$        $nf.pol$   
 number of real/complex places       $nf.r1/r2/sign$   
 discriminant of  $nf$        $nf.disc$   
 primes ramified in  $nf$        $nf.p$   
 $T_2$  matrix       $nf.t2$   
 complex roots of  $F$        $nf.roots$   
 integral basis of  $\mathbf{Z}_K$  as powers of  $\theta$        $nf.zk$   
 different/codifferent       $nf.diff, nf.codiff$   
 index  $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$        $nf.index$   
 recompute  $nf$  using current precision      `nfnewprec(nf)`  
 init relative  $rmf$   $L = K[Y]/(g)$       `rmfinit(nf, g)`  
 init  $bnf$  structure      `bnfinit(f, 1)`

### bnf members: same as $nf$ , plus

underlying  $nf$        $bnf.nf$   
 class group, regulator       $bnf.clgp, bnf.reg$   
 fundamental/torsion units       $bnf.fu, bnf.tu$   
 add  $S$ -class group and units, yield  $bnfS$

init class field structure  $bnr$       `bnrinit(bnf, m, {flag})`  
**bnr members:** same as  $bnf$ , plus  
 underlying  $bnf$        $bnr.bnf$   
 big ideal structure       $bnr.bid$   
 modulus  $m$        $bnr.mod$   
 structure of  $(\mathbf{Z}_K/m)^*$        $bnr.zkst$

## Fields, subfields, embeddings

**Defining polynomials, embeddings**  
 smallest poly defining  $f = 0$  (slow)      `polredabs(f, {flag})`  
 small poly defining  $f = 0$  (fast)      `polredbest(f, {flag})`  
 random Tschirnhausen transform of  $f$       `poltschirnhaus(f)`  
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$  ? Isomorphic?      `nfisincl(f, g), nfisisom`  
 reverse polmod  $a = A(t) \bmod T(t)$       `modreverse(a)`  
 compositum of  $\mathbf{Q}[t]/(f), \mathbf{Q}[t]/(g)$       `polcompositum(f, g, {flag})`  
 compositum of  $K[t]/(f), K[t]/(g)$       `nfcompositum(nf, f, g, {flag})`  
 splitting field of  $K$  (degree divides  $d$ )      `nfsplitting(nf, {d})`  
 signs of real embeddings of  $x$       `nfeltsign(nf, x, {pl})`  
 complex embeddings of  $x$       `nfeltembed(nf, x, {pl})`  
 $T \in K[t]$ , # of real roots of  $\sigma(T) \in R[t]$       `nfpolsturm(nf, T, {pl})`

## Subfields, polynomial factorization

subfields (of degree  $d$  of  $nf$ )      `nfsubfields(nf, {d})`  
 maximal subfields of  $nf$       `nfsubfieldsmax(nf)`  
 maximal CM subfield of  $nf$       `nfsubfieldscm(nf)`  
 $d$ -th degree subfield of  $\mathbf{Q}(\zeta_n)$       `polsubcyclo(n, d, {v})`  
 roots of unity in  $nf$       `nfroots of 1(nf)`  
 roots of  $g$  belonging to  $nf$       `nfroots(nf, g)`  
 factor  $g$  in  $nf$       `nfactor(nf, g)`

## Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       `algdep(x, k)`  
 alg. dep. with pol. coeffs for series  $s$       `seralgdep(s, x, y)`  
 small linear rel. on coords of vector  $x$       `lindep(x)`

## Basic Number Field Arithmetic (nf)

Number field elements are `t_INT, t_FRAC, t_POL, t_POLMOD`, or `t_COL` (on integral basis  $nf.zk$ ).

### Basic operations

$x + y$       `nfeltadd(nf, x, y)`  
 $x \times y$       `nfeltmul(nf, x, y)`  
 $x^n, n \in \mathbf{Z}$       `nfeltpow(nf, x, n)`  
 $x/y$       `nfeltdiv(nf, x, y)`  
 $q = x \setminus y := \text{round}(x/y)$       `nfeltdivceuc(nf, x, y)`  
 $r = x \% y := x - (x \setminus y)y$       `nfeltmod(nf, x, y)`  
 ...  $[q, r]$  as above      `nfeltdivrem(nf, x, y)`  
 reduce  $x$  modulo ideal  $A$       `nfeltreduce(nf, x, A)`  
 absolute trace  $\text{Tr}_{K/\mathbf{Q}}(x)$       `nfelttrace(nf, x)`  
 absolute norm  $N_{K/\mathbf{Q}}(x)$       `nfeltnorm(nf, x)`

### Multiplicative structure of $K^*$ ; $K^*/(K^*)^n$

valuation  $v_p(x)$       `nfeltval(nf, x, p)`  
 ... write  $x = \pi^{v_p(x)}y$       `nfeltval(nf, x, p, &y)`  
 quadratic Hilbert symbol (at  $p$ )      `nfhilbert(nf, a, b, {p})`  
 $b$  such that  $xb^n = v$  is small      `idealredmodpower(nf, x, n)`

## Maximal order and discriminant

integral basis of field  $\mathbf{Q}[x]/(f)$       `nfbasis(f)`  
 field discriminant of  $\mathbf{Q}[x]/(f)$       `nfdisc(f)`  
 ... and factorization      `nfdiscfactors(f)`  
 express  $x$  on integer basis      `nfalgtobasis(nf, x)`  
 express element  $x$  as a polmod      `nfbasistoalg(nf, x)`

## Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w, |\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).

$\zeta_K$  as Dirichlet series,  $N(I) < b$       `dirzetak(nf, b)`  
 init  $\zeta_K^{(k)}(s)$  for  $k \leq n$       `L = lfuninit(bnf, R, {n = 0})`  
 compute  $\zeta_K(s)$  ( $n$ -th derivative)      `lfun(L, s, {n = 0})`  
 compute  $\Lambda_K(s)$  ( $n$ -th derivative)      `lfunlambda(L, s, {n = 0})`

init  $\zeta_K^{(k)}(s, \chi)$  for  $k \leq n$       `L = lfuninit([bnr, chi], R, {n = 0})`  
 compute  $L_K(s, \chi)$  ( $n$ -th derivative)      `lfun(L, s, {n})`  
 Artin root number of  $K$       `bnrrootnumber(bnr, chi, {flag})`  
 $L(1, \chi)$ , for all  $\chi$  trivial on  $H$       `bnrL1(bnr, {H}, {flag})`

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually  $bnr$  (ray class field),  $bnr, H$  (congruence subgroup) or  $bnr, \chi$  (character on `bnr.clgp`). Any of these define a unique abelian extension of  $K$ .

units /  $S$ -units      `bnfunits(bnf, {S})`  
 remove GRH assumption from  $bnf$       `bnfcertify(bnf)`  
 expo. of ideal  $x$  on class gp      `bnfisprincipal(bnf, x, {flag})`  
 expo. of ideal  $x$  on ray class gp      `bnrisprincipal(bnr, x, {flag})`  
 expo. of  $x$  on fund. units      `bnfisunit(bnf, x)`  
 ... on  $S$ -units,  $U$  is `bnfunits(bnf, S)`      `bnfisunit(bnfs, x, U)`  
 signs of real embeddings of  $bnf.fu$       `bnfsignunit(bnf)`  
 narrow class group      `bnfnarrow(bnf)`

## Class Field Theory

ray class number for modulus  $m$       `bnrclassno(bnf, m)`  
 discriminant of class field      `bnrdisc(a1, {a2})`  
 ray class numbers,  $l$  list of moduli      `bnrclassnolist(bnf, l)`  
 discriminants of class fields      `bnrdisclist(bnf, l, {arch}, {flag})`  
 decode output from `bnrdisclist`      `bnfdecodemodule(nf, fa)`  
 is modulus the conductor?      `bnrisconductor(a1, {a2})`  
 is class field ( $bnr, H$ ) Galois over  $K^G$       `bnrisgalois(bnr, G, H)`  
 action of automorphism on `bnr.gen`      `bnrgaloismatrix(bnr, aut)`  
 apply `bnrgaloismatrix M` to  $H$       `bnrgaloisapply(bnr, M, H)`  
 characters on `bnr.clgp` s.t.  $\chi(g_i) = e(v_i)$       `bnrchar(bnr, g, {v})`  
 conductor of character  $\chi$       `bnrconductor(bnr, chi)`  
 conductor of extension      `bnrconductor(a1, {a2}, {flag})`  
 conductor of extension  $K[Y]/(g)$       `rmfconductor(bnf, g)`  
 canonical projection  $\text{Cl}_F \rightarrow \text{Cl}_f, f | F$       `bnrmap`  
 Artin group of extension  $K[Y]/(g)$       `rnfnormgroup(bnr, g)`  
 subgroups of  $bnr$ , index  $\leq b$       `subgrouplist(bnr, b, {flag})`  
 class field defined by  $H < \text{Cl}_f$       `bnrclassfield(bnr, H)`  
 ... low level equivalent, prime degree      `rnfkummer(bnr, H)`  
 same, using Stark units (real field)      `bnrstark(bnr, sub, {flag})`  
 is a an  $n$ -th power in  $K_v$  ?      `nfislocalpower(nf, v, a, n)`  
 cyclic  $L/K$  satisf. local conditions      `nfgrunwaldwang(nf, P, D, pl)`

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## Logarithmic class group

logarithmic $\ell$ -class group	<code>bnflog(bnf, \ell)</code>
$[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$	<code>bnflogef(bnf, pr)</code>
$\exp \deg_F(A)$	<code>bnflogdegree(bnf, A, \ell)</code>
is $\ell$ -extension $L/K$ locally cyclotomic	<code>rnfislocalcyclo(rnf)</code>

## Ideals: elements, primes, or matrix of generators in HNF

is $id$ an ideal in $nf$ ?	<code>nfisideal(nf, id)</code>
is $x$ principal in $bnf$ ?	<code>bnfisprincipal(bnf, x)</code>
give $[a, b]$ , s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$	<code>idealtwoelt(nf, x, \{a\})</code>
put ideal $a$ ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form	<code>idealhnf(nf, a, \{b\})</code>
norm of ideal $x$	<code>idealnrm(nf, x)</code>
minimum of ideal $x$ (direction $v$ )	<code>idealmin(nf, x, v)</code>
LLL-reduce the ideal $x$ (direction $v$ )	<code>idealred(nf, x, \{v\})</code>

## Ideal Operations

add ideals $x$ and $y$	<code>idealadd(nf, x, y)</code>
multiply ideals $x$ and $y$	<code>idealmul(nf, x, y, \{flag\})</code>
intersection of ideal $x$ with $Q$	<code>idealdown(nf, x)</code>
intersection of ideals $x$ and $y$	<code>idealintersect(nf, x, y, \{flag\})</code>
$n$ -th power of ideal $x$	<code>idealpwn(nf, x, n, \{flag\})</code>
inverse of ideal $x$	<code>idealinv(nf, x)</code>
divide ideal $x$ by $y$	<code>idealdiv(nf, x, y, \{flag\})</code>
Find $(a, b) \in x \times y, a + b = 1$	<code>idealaddtoone(nf, x, \{y\})</code>
coprime integral $A, B$ such that $x = A/B$	<code>idealnumden(nf, x)</code>

## Primes and Multiplicative Structure

check whether $x$ is a maximal ideal	<code>idealismaximal(nf, x)</code>
factor ideal $x$ in $\mathbf{Z}_K$	<code>idealfactor(nf, x)</code>
expand ideal factorization in $K$	<code>idealfactorback(nf, f, \{e\})</code>
is ideal $A$ an $n$ -th power ?	<code>idealispower(nf, A, n)</code>
expand elt factorization in $K$	<code>nffactorback(nf, f, \{e\})</code>
decomposition of prime $p$ in $\mathbf{Z}_K$	<code>idealprimedec(nf, p)</code>
valuation of $x$ at prime ideal $pr$	<code>idealval(nf, x, pr)</code>
weak approximation theorem in $nf$	<code>idealchinese(nf, x, y)</code>
$a \in K$ , s.t. $v_p(a) = v_p(x)$ if $v_p(x) \neq 0$	<code>idealappr(nf, x)</code>
$a \in K$ such that $(a \cdot x, y) = 1$	<code>idealcoprime(nf, x, y)</code>
give $bid = \text{structure of } (\mathbf{Z}_K/id)^*$	<code>idealstar(nf, id, \{flag\})</code>
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$	<code>idealprincipalunits(nf, pr, k)</code>
discrete log of $x$ in $(\mathbf{Z}_K/bid)^*$	<code>ideallog(nf, x, bid)</code>
<code>idealstar</code> of all ideals of norm $\leq b$	<code>ideallist(nf, b, \{flag\})</code>
add Archimedean places	<code>ideallistarch(nf, b, \{ar\}, \{flag\})</code>
init <code>modpr</code> structure	<code>nfmodprinit(nf, pr, \{v\})</code>
project $t$ to $\mathbf{Z}_K/pr$	<code>nfmodpr(nf, t, modpr)</code>
lift from $\mathbf{Z}_K/pr$	<code>nfmodprlift(nf, t, modpr)</code>

## Galois theory over $\mathbf{Q}$

conjugates of a root $\theta$ of $nf$	<code>nfgaloisconj(nf, \{flag\})</code>
apply Galois automorphism $s$ to $x$	<code>nfgaloisapply(nf, s, x)</code>
Galois group of field $\mathbf{Q}[x]/(f)$	<code>polgalois(f)</code>
initializes a Galois group structure $G$	<code>galoisinit(pol, \{den\})</code>
character table of $G$	<code>galoischartable(G)</code>
conjugacy classes of $G$	<code>galoisconjclasses(G)</code>
$\det(1 - \rho(g)T)$ , $\chi$ character of $\rho$	<code>galoischarpoly(G, \chi, \{o\})</code>
$\det(\rho(g))$ , $\chi$ character of $\rho$	<code>galoischarpoly(G, \chi, \{o\})</code>
action of $p$ in $nfgaloisconj$ form	<code>galoispermopol(G, \{p\})</code>
identify as abstract group	<code>galoisidentify(G)</code>
export a group for GAP/MAGMA	<code>galoisexport(G, \{flag\})</code>
subgroups of the Galois group $G$	<code>galoissubgroups(G)</code>
is subgroup $H$ normal?	<code>galoisnormal(G, H)</code>

subfields from subgroups	<code>galoisubfields(G, \{flag\}, \{v\})</code>
fixed field	<code>galoisfixedfield(G, perm, \{flag\}, \{v\})</code>
Frobenius at maximal ideal $P$	<code>idealfrobenius(nf, G, P)</code>
ramification groups at $P$	<code>idealarmgroups(nf, G, P)</code>
is $G$ abelian?	<code>galoisisabelian(G, \{flag\})</code>
abelian number fields/ $\mathbf{Q}$	<code>galoissubcyclo(N, H, \{flag\}, \{v\})</code>

## The galpol package

query the package: polynomial	<code>galoisgetpol(a, b, \{s\})</code>
$\dots$ : permutation group	<code>galoisgetgroup(a, b)</code>
$\dots$ : group description	<code>galoisgetname(a, b)</code>

## Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $T \in K[x]$ .

absolute equation of $L$	<code>rnfequation(nf, T, \{flag\})</code>
is $L/K$ abelian?	<code>rnfisabelian(nf, T)</code>
relative <code>nfaltobasis</code>	<code>rnfaltobasis(rnf, x)</code>
relative <code>nfbasistoalg</code>	<code>rnfbasistoalg(rnf, x)</code>
relative <code>idealhnf</code>	<code>rnfidealhnf(rnf, x)</code>
relative <code>idealmul</code>	<code>rnfidealmul(rnf, x, y)</code>
relative <code>idealtwoelt</code>	<code>rnfidealtwoelt(rnf, x)</code>

## Lifts and Push-downs

absolute $\rightarrow$ relative representation for $x$	<code>rnfeltabstorel(rnf, x)</code>
relative $\rightarrow$ absolute representation for $x$	<code>rnfeltreltoabs(rnf, x)</code>
lift $x$ to the relative field	<code>rnfeltup(rnf, x)</code>
push $x$ down to the base field	<code>rnfeltdown(rnf, x)</code>
idem for $x$ ideal: <code>(rnfideal)reltoabs, abstorel, up, down</code>	

## Norms and Trace

relative norm of element $x \in L$	<code>rnfeltnorm(rnf, x)</code>
relative trace of element $x \in L$	<code>rnfeltrace(rnf, x)</code>
absolute norm of ideal $x$	<code>rnfidealnrmabs(rnf, x)</code>
relative norm of ideal $x$	<code>rnfidealnrmrel(rnf, x)</code>
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$	<code>bnfisintnorm(bnf, x)</code>
is $x \in \mathbf{Q}$ a norm from $K$ ?	<code>bnfisnorm(bnf, x, \{flag\})</code>
initialize $T$ for norm eq. solver	<code>rnfisnorminit(K, pol, \{flag\})</code>
is $a \in K$ a norm from $L$ ?	<code>rnfisnorm(T, a, \{flag\})</code>
initialize $t$ for Thue equation solver	<code>thueinit(f)</code>
solve Thue equation $f(x, y) = a$	<code>thue(t, a, \{sol\})</code>
characteristic poly. of $a \bmod T$	<code>rnfcharpoly(nf, T, a, \{v\})</code>

## Factorization

factor ideal $x$ in $L$	<code>rnfidealfactor(rnf, x)</code>
$[S, T]: T_{i,j} \mid S_i; S$ primes of $K$ above $p$	<code>rnfidealprimedec(rnf, p)</code>

## Maximal order $\mathbf{Z}_L$ as a $\mathbf{Z}_K$ -module

relative <code>polredbest</code>	<code>rnfpolredbest(nf, T)</code>
relative <code>polredabs</code>	<code>rnfpolredabs(nf, T)</code>
relative Dedekind criterion, prime $pr$	<code>rnfddedekind(nf, T, pr)</code>
discriminant of relative extension	<code>rnfdisc(nf, T)</code>
pseudo-basis of $\mathbf{Z}_L$	<code>rnfpsudobasis(nf, T)</code>

## General $\mathbf{Z}_K$ -modules: $M = [\text{matrix, vec. of ideals}] \subset L$

relative HNF / SNF	<code>nfhnf(nf, M), nfnfnf</code>
multiple of $\det M$	<code>nfdetint(nf, M)</code>
HNF of $M$ where $d = nfdetint(M)$	<code>nfhnfmod(x, d)</code>
reduced basis for $M$	<code>rnflllgram(nf, T, M)</code>
determinant of pseudo-matrix $M$	<code>rnfdet(nf, M)</code>
Steinitz class of $M$	<code>rnfsteinitz(nf, M)</code>

$\mathbf{Z}_K$ -basis of  $M$  if  $\mathbf{Z}_K$ -free, or  $0$   
 $n$ -basis of  $M$ , or  $(n + 1)$ -generating set  
 is  $M$  a free  $\mathbf{Z}_K$ -module?

<code>rnfnhbasis(bnf, M)</code>
<code>rnfbasis(bnf, M)</code>
<code>rnfisfree(bnf, M)</code>

## Associative Algebras

$A$  is a general associative algebra given by a multiplication table  $mt$  (over  $\mathbf{Q}$  or  $\mathbf{F}_p$ ); represented by  $al$  from `algtableinit`.  
 create  $al$  from  $mt$  (over  $\mathbf{F}_p$ ) `algtableinit(mt, \{p = 0\})`  
 group algebra  $\mathbf{Q}[G]$  (or  $\mathbf{F}_p[G]$ ) `alggroup(G, \{p = 0\})`  
 center of group algebra `alggroupcenter(G, \{p = 0\})`

## Properties

is $(mt, p)$ OK for <code>algtableinit</code> ?	<code>algisassociative(mt, \{p = 0\})</code>
multiplication table $mt$	<code>algmtable(al)</code>
dimension of $A$ over prime subfield	<code>algdim(al)</code>
characteristic of $A$	<code>algchar(al)</code>
is $A$ commutative?	<code>algiscommutative(al)</code>
is $A$ simple?	<code>algissimple(al)</code>
is $A$ semi-simple?	<code>algissemisimple(al)</code>
center of $A$	<code>algcenter(al)</code>
Jacobson radical of $A$	<code>algradical(al)</code>
radical $J$ and simple factors of $A/J$	<code>algsimpledec(al)</code>

## Operations on algebras

create $A/I$ , $I$ two-sided ideal	<code>algquotient(al, I)</code>
create $A_1 \otimes A_2$	<code>algtensor(al1, al2)</code>
create subalgebra from basis $B$	<code>algsubalg(al, B)</code>
quotients by ortho. central idempotents $e$	<code>algcentproj(al, e)</code>
isomorphic alg. with integral mult. table	<code>algmakeintegral(mt)</code>
prime subalgebra of semi-simple $A$ over $\mathbf{F}_p$	<code>algprimesubalg(al)</code>
find isomorphism $A \cong M_d(\mathbf{F}_q)$	<code>algsplit(al)</code>

## Operations on lattices in algebras

lattice generated by cols. of $M$	<code>alglathnf(al, M)</code>
$\dots$ by the products $xy, x \in lat1, y \in lat2$	<code>alglatmul(al, lat1, lat2)</code>
sum $lat1 + lat2$ of the lattices	<code>alglatadd(al, lat1, lat2)</code>
intersection $lat1 \cap lat2$	<code>alglatinter(al, lat1, lat2)</code>
test $lat1 \subset lat2$	<code>alglatsubset(al, lat1, lat2)</code>
generalized index $(lat2 : lat1)$	<code>alglatindex(al, lat1, lat2)</code>
$\{x \in al \mid x \cdot lat1 \subset lat2\}$	<code>alglatlefttransporter(al, lat1, lat2)</code>
$\{x \in al \mid lat1 \cdot x \subset lat2\}$	<code>alglatrighttransporter(al, lat1, lat2)</code>
test $x \in lat$ (set $c = \text{coord. of } x$ )	<code>alglatcontains(al, lat, x, \{\&amp;c\})</code>
element of $lat$ with coordinates $c$	<code>alglatelement(al, lat, c)</code>

## Operations on elements

$a + b, a - b, -a$	<code>algadd(al, a, b), algsub, algneg</code>
$a \times b, a^2$	<code>algmul(al, a, b), algsqrt</code>
$a^n, a^{-1}$	<code>algpow(al, a, n), alginv</code>
is $x$ invertible? (then set $z = x^{-1}$ )	<code>algisin(al, x, \{\&amp;z\})</code>
find $z$ such that $x \times z = y$	<code>algdivl(al, x, y)</code>
find $z$ such that $z \times x = y$	<code>algdivr(al, x, y)</code>
does $s$ s.t. $x \times z = y$ exist? (set it)	<code>algisdivl(al, x, y, \{\&amp;z\})</code>
matrix of $v \mapsto x \cdot v$	<code>algtomatrix(al, x)</code>
absolute norm	<code>algnorm(al, x)</code>
absolute trace	<code>algtrace(al, x)</code>
absolute char. polynomial	<code>algcharpoly(al, x)</code>
given $a \in A$ and polynomial $T$ , return $T(a)$	<code>algpoleval(al, T, a)</code>
random element in a box	<code>algramdom(al, b)</code>

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## Central Simple Algebras

$A$  is a central simple algebra over a number field  $K$ ; represented by  $al$  from **alginit**;  $K$  is given by a  $nf$  structure.

create CSA from data      **alginit**( $B, C, \{v\}, \{maxord = 1\}$ )  
multiplication table over  $K$        $B = K, C = mt$   
cyclic algebra ( $L/K, \sigma, b$ )       $B = rnf, C = [\sigma, b]$   
quaternion algebra ( $a, b$ ) $_K$        $B = K, C = [a, b]$   
matrix algebra  $M_d(K)$        $B = K, C = d$   
local Hasse invariants over  $K$        $B = K, C = [d, [PR, HF], HI]$

### Properties

type of  $al$  ( $mt, CSA$ )      **algtype**( $al$ )  
dimension of  $A$  over  $\mathbf{Q}$       **algdim**( $al, 1$ )  
dimension of  $al$  over its center  $K$       **algdim**( $al$ )  
degree of  $A$  ( $= \sqrt{\dim_K A}$ )      **algdegree**( $al$ )  
 $al$  a cyclic algebra ( $L/K, \sigma, b$ ); return  $\sigma$       **algaut**( $al$ )  
... return  $b$       **algb**( $al$ )  
... return  $L/K$ , as an  $rnf$       **algsplittingfield**( $al$ )  
split  $A$  over an extension of  $K$       **algsplittingdata**( $al$ )  
splitting field of  $A$  as an  $rnf$  over center      **algsplittingfield**( $al$ )  
multiplication table over center      **algremltable**( $al$ )  
places of  $K$  at which  $A$  ramifies      **algramifiedplaces**( $al$ )  
Hasse invariants at finite places of  $K$       **alghassef**( $al$ )  
Hasse invariants at infinite places of  $K$       **alghassei**( $al$ )  
Hasse invariant at place  $v$       **alghasse**( $al, v$ )  
index of  $A$  over  $K$  (at place  $v$ )      **algindex**( $al, \{v\}$ )  
is  $al$  a division algebra? (at place  $v$ )      **algisdivision**( $al, \{v\}$ )  
is  $A$  ramified? (at place  $v$ )      **algisramified**( $al, \{v\}$ )  
is  $A$  split? (at place  $v$ )      **algissplit**( $al, \{v\}$ )

### Operations on elements

reduced norm      **algnorm**( $al, x$ )  
reduced trace      **algtrace**( $al, x$ )  
reduced char. polynomial      **algcharpoly**( $al, x$ )  
express  $x$  on integral basis      **algalgtobasis**( $al, x$ )  
convert  $x$  to algebraic form      **algbasistoalg**( $al, x$ )  
map  $x \in A$  to  $M_d(L)$ ,  $L$  split. field      **algtomatrix**( $al, x$ )

### Orders

**Z**-basis of order  $\mathcal{O}_0$       **algbasis**( $al$ )  
discriminant of order  $\mathcal{O}_0$       **algdisc**( $al$ )  
**Z**-basis of natural order in terms  $\mathcal{O}_0$ 's basis      **alginvbasis**( $al$ )

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